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The theory of prime ideals of Leavitt path algebras over arbitrary graphs

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ABSTRACT

Given an arbitrary graph E and a field K, the prime ideals as well as the primitive ideals of the Leavitt path algebra $L_K(E)$ are completely characterized in terms of their generators. The stratification of the prime spectrum of $L_K(E)$ is indicated with information on its individual stratum. Necessary and sufficient conditions are given on the graph E under which every prime ideal of the Leavitt path algebra $L_K(E)$ is primitive. Leavitt path algebras with Krull dimension zero are characterized and those with various prescribed Krull dimension are constructed. The minimal prime ideals of $L_K(E)$ are described in terms of the graphical properties of E and using this, complete descriptions of the height one as well as the co-height one prime ideals of $L_K(E)$ are given.

1. Introduction

The Leavitt path algebras were introduced in [1] and [9] as algebraic analogs of the C^*-algebras [19] and the study of their algebraic structure has been the subject of a series of papers in recent years (see, for e.g., [1–13,15,22]). In this paper we develop the theory of the prime ideals of the Leavitt path algebras $L_K(E)$ for an arbitrary-sized graph E. Here the graph E is arbitrary in the sense that no restriction is placed either on the number of vertices in E or on the number of edges emitted by a single vertex in E (such as row-finite or countable). We first give complete characterizations, in terms of the generators, of the prime ideals as well as the primitive ideals of the Leavitt path algebra $L_K(E)$. We also describe the stratification of the prime spectrum $Spec(L_K(E))$ with information about its individual stratum many of which are homeomorphic to $Spec(K[x,x^{-1}])$. Our investigation shows that the non-graded prime ideals of a Leavitt path algebra $L_K(E)$ are always primitive and, as

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also noted in [11], there are graded prime ideals of $L_K(E)$ which are not primitive. This is in contrast to the situation for the graph C^*-algebras in which there is no distinction between the prime and the primitive ideals (see [14,16,21]). Using our characterization of right (= left) primitive ideals of a Leavitt path algebra, we give necessary and sufficient conditions on the graph E under which every prime ideal of $L_K(E)$ is primitive. Examples are constructed to illustrate the different possibilities for the prime and the primitive spectrum of $L_K(E)$. In the case when E is a row-finite graph, it was shown in [11] that a bijection exists between the set of prime ideals of the Leavitt path algebra $L_K(E)$ and a certain set which involves subsets of vertices (called maximal tails) and the prime spectrum of $K[x, x^{-1}]$. This was analogous to the work done in [16] for graph C^*-algebras. In this paper, we extend and sharpen the bijective correspondence of [11] to the case when E is an arbitrary-sized graph. There are a number of applications of our main result (Theorem 3.12). First we describe those graphs E for which every non-zero prime ideal of $L_K(E)$ is maximal and this leads to the characterization the Leavitt path algebras whose Krull dimension is zero. Examples are constructed of Leavitt path algebras $L_K(E)$ with various prescribed finite or infinite Krull dimension. As another application, we show that a graph E satisfies the Condition (K) if and only if every prime ideal of $L_K(E)$ is graded. Next we describe the minimal prime ideals of a Leavitt path algebra. The height 1 prime ideals play an important part in the study of commutative rings and algebraic geometry. As a first step to examine their role in the study of Leavitt path algebras, we characterize not only the height 1 prime ideals but also the co-height 1 prime ideals of $L_K(E)$ in terms of the graphical properties of E. Finally, we also consider when a prime homomorphic image of a Leavitt path algebra is again a Leavitt path algebra.

2. Preliminaries

For the general notation and terminology used in this paper, we refer to [1,11,21]. We give a short description of some of the concepts that we will be using. A directed graph $E = (E^0, E^1, r, s)$ consists of a set E^0 of vertices, a set E^1 of edges and maps r, s from E^1 to E^0. For each $e \in E^1$, $s(e) = u$ is called the *source* of e and $r(e) = v$ the *range* of e and e is called an edge from u to v. All the graphs that we consider in this note are arbitrary. A vertex v is called a *sink* if it emits no edges. It is called a *regular vertex* if $0 < |s^{-1}(v)| < \infty$. If $|s^{-1}(v)| = \infty$, we say v is an *infinite emitter*.

Given a graph E, $(E^1)^*$ denotes the set of symbols e^*, one for each $e \in E^1$, called the ghost edges.

Definition. Let E be a directed graph and K be any field. The *Leavitt path algebra* $L_K(E)$ of the graph E with coefficients in K is the K-algebra generated by a set $\{v\colon\ v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^*\colon\ e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.
(2) $r(e)e^* = e^* = e^*s(e)\infty$ for all $e \in E^1$.
(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1,\, s(e)=v} ee^*.$$

A *path* μ is a finite sequence of edges $e_1 \dots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \dots, n-1$. μ^* denotes the corresponding ghost path $e_n^* \dots e_1^*$. The path μ is said to be a *closed path* if $r(e_n) = s(e_1)$ and in this case, μ is said to be *based* at $s(e_1)$. A closed path $\mu = e_1 \dots e_n$ is said to be *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \dots, n$. The closed path μ is called a *cycle* if it does not pass through any vertex twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

An edge f is called an *exit* to a path $e_1 \dots e_n$ if there is an i such that $s(f) = s(e_i)$ and $f \neq e_i$. A graph E satisfies *Condition* (L) provided every simple closed path in E has an exit, or equivalently, every cycle in E has an exit. The graph E is said to satisfy the *Condition* (K) provided no vertex v in

E is the base of precisely one simple closed path, that is, either no simple closed path is based at v or at least two are based at v.

Every element in a Leavitt path algebra $L_K(E)$ can be written in the form $\sum_{i=1}^{n} k_i\alpha_i\beta_i^*$ where $k_i \in K$, n a positive integer and α, β are paths in E (see [1]).

The following well-known result turns out to be very useful in our investigation: If there exists a cycle c without exits and is based at a vertex v in a graph E, then the subring $vL_K(E)v \cong K[x, x^{-1}]$ under an isomorphism that maps v to 1, c to x and c^* to x^{-1}. Indeed this isomorphism is evident if one notices that a typical element of $vL_K(E)v$ is a K-linear combination of elements of the form $v\alpha\beta^*v$ which simplifies, since c has no exits, to v, c^n or $(c^*)^n$ for some integer n.

If there is a path from a vertex u to a vertex v, we write $u \geqslant v$. A subset H of vertices is called a *hereditary* set if whenever $u \in H$ and $u \geqslant v$ for some vertex v, then $v \in H$. A set of vertices H is said to be *saturated* if, for any regular vertex v, $r(s^{-1}(v)) \subset H$ implies $v \in H$. If I is a two-sided ideal of $L_K(E)$, it is easy to see that $I \cap E^0$ is a hereditary saturated subset of E^0.

For every non-empty subset X of vertices in a graph E, we can define *the restricted subgraph* E_X where $(E_X)^0 = X$ and $(E_X)^1 = \{e \in E^1\colon\ s(e), r(e) \in X\}$.

We shall be making extensive use of the following concepts and results from [22]. A vertex w is called a *breaking vertex* of a hereditary saturated subset H if $w \in E^0 \backslash H$ is an infinite emitter with the property that $1 \leqslant |s^{-1}(v) \cap r^{-1}(E^0 \backslash H)| < \infty$. The set of all breaking vertices of H is denoted by B_H. For any $v \in B_H$, v^H denotes the element $v - \sum_{s(e)=v,\, r(e)\notin H} ee^*$. Given a hereditary saturated subset H and a subset $S \subset B_H$, (H, S) is called an admissible pair and $I_{(H,S)}$ denotes the ideal generated by $H \cup \{v^H\colon\ v \in S\}$. It was shown in [22] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I_{(H,S)}$ for some admissible pair (H, S). Moreover, it was shown that $I_{(H,S)} \cap E^0 = H$ and $\{v \in B_H\colon\ v^H \in I\} = S$.

Given an admissible pair (H, S), the corresponding *Quotient Graph* $E\backslash(H, S)$ is defined as follows:

$$\big(E\backslash(H,S)\big)^0 = \big(E^0\backslash H\big) \cup \big\{v'\colon\ v \in B_H\backslash S\big\};$$
$$\big(E\backslash(H,S)\big)^1 = \big\{e \in E^1\colon\ r(e) \notin H\big\} \cup \big\{e'\colon\ e \in E^1,\ r(e) \in B_H\backslash S\big\}.$$

Further r and s are extended to $(E\backslash(H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$.

Notice that the elements v', where $v \in B_H\backslash S$, are all sinks in the graph $E\backslash(H, S)$. Theorem 5.7 of [22] states that there is an epimorphism $\phi : L_K(E) \to L_K(E\backslash(H, S))$ with $\ker\phi = I_{(H,S)}$ and that $\phi(v^H) = v'$ for $v \in B_H\backslash S$. Thus $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H, S))$. (This theorem has been established in [22] under the hypothesis that E is a graph with at most countably many vertices and edges; however, an examination of the proof reveals that the countability condition on E is not utilized. So Theorem 5.7 of [22] holds for arbitrary graphs E.)

Let H be a non-empty hereditary saturated set of vertices in a graph E. A subset M of H is said to be a *maximal tail* in H (see [9,14]) if it satisfies the following conditions:

(MT-1) If $v \in M$ and $u \in H$ with $u \geqslant v$, then $u \in M$;
(MT-2) If $v \in M$ is a regular vertex, then there is an $e \in E^1$ with $s(e) = v$ and $r(e) \in M$;
(MT-3) For any two $u, v \in M$ there exists $w \in M$ such that $u \geqslant w$ and $v \geqslant w$.

It is easy to see that $M \subset E^0$ satisfies both the MT-1 and the MT-2 conditions if and only if $E^0\backslash M$ is a hereditary saturated subset of E^0.

Given a vertex v in a graph E, we attach two special sets of vertices.

Define $T(v) = \{w \in E^0\colon\ v \geqslant w\}$ and $M(v) = \{w \in E^0\colon\ w \geqslant v\}$.

The set $T(v)$ (called the *tree of* v in the literature) is the smallest hereditary set containing v. The set $M(v)$ is maximal tail containing v whenever v is a sink, an infinite emitter or a regular vertex lying on a cycle in $M(v)$. But when v is a regular vertex not lying on a cycle, $M(v)$ satisfies the MT-1 and MT-3 conditions, but may not satisfy the MT-2 condition as is clear by considering the vertex v in the graph $\underset{u}{\circ} \to \underset{w}{\circ} \to \underset{v}{\circ} \to \underset{w'}{\circ}$.

In this paper "ideal" means "two-sided ideal". An ideal P of a ring R is called a left (right) *primitive ideal* if it is the left (right) annihilator of a simple left (right) R-module. A ring R is called a left (right) *primitive ring* if $\{0\}$ is a right (left) primitive ideal.

Recall that an ideal P of a ring R is called a *prime ideal* if, given any ideals A, B of R, $AB \subset P$ implies that either $A \subset P$ or $B \subset P$. If R is a graded ring, graded by a group (such as the Leavitt path algebra $L_K(E)$), then it was shown in [20, Proposition II.1.4], that a graded ideal P will be a prime ideal, if P satisfies the above property only for graded ideals A and B. This observation will be used in the sequel.

3. Prime ideals of $L_K(E)$

In this section, we give a complete characterization of the prime ideals of a Leavitt path algebra $L_K(E)$ of an arbitrary graph E. In [11] graded prime ideals of $L_K(E)$ were described for row-finite graphs E. We extend this result to the case when E is an arbitrary graph. Moreover, we also characterize the non-graded prime ideals of $L_K(E)$ for arbitrary-sized graphs E by means of their generating sets. As will be clear from subsequent sections, this helps in our deeper study of the prime ideals of Leavitt path algebras. The bijective correspondence established in [11] for prime ideals of $L_K(E)$ for a row-finite graph E is also extended to the case when E is an arbitrary graph.

We shall be using the following description of a prime Leavitt path algebra $L_K(E)$ that was established for row-finite graphs E in [11] and for arbitrary graphs E in [5]. I understand that M. Siles Molina also has independently obtained the following characterization.

Theorem 3.1. *(See [5,11].) Let E be an arbitrary graph and K be any field. Then the Leavitt path algebra $L_K(E)$ is a prime ring if and only if E^0 satisfies the* MT-3 *condition.*

The following theorem from [6] is used in the sequel.

Theorem 3.2. *(See [6].) Let E be an arbitrary graph and K be any field. Then any non-zero ideal of the $L_K(E)$ is generated by elements of the form*

$$\left(u + \sum_{i=1}^{k} k_i g^{r_i}\right)\left(u - \sum_{e\in X} ee^*\right)$$

where $u \in E^0$, $k_i \in K$, r_i are positive integers, X is a finite (possibly empty) proper subset of $s^{-1}(u)$ and, whenever $k_i \neq 0$ for some i, then g is a unique cycle based at u.

Before proving our main theorem, we shall establish a series of useful lemmas. The next lemma specializes Theorem 3.2 to the case of ideals containing no vertices.

Lemma 3.3. *Suppose E is an arbitrary graph and K is any field. If N is a non-zero ideal of $L_K(E)$ which does not contain any vertices of E, then N is a non-graded ideal generated by elements of the form $y = (u + \sum_{i=1}^n k_i g^{r_i})$ where g is a (unique) cycle without exits based at a vertex u and $k_i \in K$ with at least one $k_i \neq 0$.*

Proof. Since N is non-zero and $H = N \cap E^0 = \Phi$, N must be a non-graded ideal, because if N was a graded ideal, then by Tomforde [22], $N = I_{(H,S)}$ which must be $\{0\}$, as $H = \Phi$ and $S = N \cap B_H = \Phi$. From Theorem 3.2, we know that N is generated by elements of the form $y = (u + \sum_{i=1}^n k_i g^{r_i})(u - \sum_{e\in X} ee^*) \neq 0$ where g is a cycle in E based at a vertex u and where X is a finite proper subset of $s^{-1}(u)$.

Our first step is to show that the cycle g has no exits and that X must be an empty set. Suppose $f \in s^{-1}(u)\backslash X$ with $r(f) = w$. If f is not the initial edge of g, then $f^*g = 0$ and $f^*yf = f^*(u + \sum_{i=1}^n k_i g^{r_i})f = f^*uf = r(f) = w \in N$, contradicting the fact that N contains no vertices. If f is the initial edge of g say $g = f\alpha$, then $f^*yf = w + \sum_{i=1}^n k_i h^{r_i} \in N$ where $w = r(f)$ and h is the cycle αf. Then $\alpha^*(w+\sum_{i=1}^n k_i h^{r_i})\alpha = u+\sum_{i=1}^n k_i g^{r_i} \in N$. If there is an exit e at a vertex u' on g and if β is the part of g connecting u to u' (where we take $\beta = u$ if $u' = u$) and γ is the part of g from u' to u (so that $g = \beta\gamma$), then, denoting $\gamma\beta$ by d, we get $e^*\beta^*(u+\sum_{i=1}^n k_i g^{r_i})\beta e = e^*(u'+\sum_{i=1}^n k_i d^{r_i})e = r(e) \in N$, a contradiction. Thus the cycle g has no exits. In particular, $|s^{-1}(u)| = 1$ and this implies, as $y \neq 0$, that X is an empty set. Thus the generators of N are of the form $y = (u+\sum_{i=1}^n k_i g^{r_i})$. $\square$

As an immediate consequence of the preceding lemma (where we get cycles g without exits), we obtain the following well-known result established in [12].

Corollary 3.4. *Let E be an arbitrary graph. If E satisfies Condition* (L)*, then every non-zero two-sided ideal of $L_K(E)$ contains a vertex.*

Lemma 3.5. *Let E be an arbitrary graph and K be any field. Suppose E^0 satisfies the* MT-3 *condition. If N is a non-zero ideal of $L_K(E)$ which does not contain any vertices of E, then there is a unique cycle c without exits in E and N is a non-graded principal ideal generated by $p(c)$, where $p(x)$ is a polynomial belonging to $K[x]$.*

Proof. By Lemma 3.3, N is non-graded ideal generated by elements of the form $y = (u+\sum_{i=1}^n k_i g^{r_i})$ where g is a cycle without exits based at u. Now the MT-3 condition on E^0 implies that g is the only cycle without exits in E (except possibly a permutation of its vertices). This means that if there is a cycle c without exits based at a vertex v in E, the cycle g based at u is the same as the cycle c based at v, obtained possibly by a rotation of the vertices on c. If α is the part of c from v to u and β is the part of c from u to v (so that $c = \alpha\beta$ and $g = \beta\alpha$), then $\beta^*(u+\sum_{i=1}^n k_i g^{r_i})\beta = v+\sum_{i=1}^n k_i c^{r_i} \in N$ and $\alpha^*(v+\sum_{i=1}^n k_i c^{r_i})\alpha = u+\sum_{i=1}^n k_i g^{r_i} = y \in N$. From this it is clear that we can select a generating set for the ideal N consisting of elements of the form $(v+\sum_{i=1}^n k_i c^{r_i})$ with this fixed cycle c based at v, where $k_i \neq 0$ for at least one i. We shall denote the element $(v+\sum_{i=1}^n k_i c^{r_i})$ by $f(c)$, where $f(x) = x^0 + \sum_{i=1}^n k_i x^{r_i} \in K[x]$ is a polynomial of positive degree (and where we use the convention that $c^0 = v$). Let $p(x)$ be a polynomial of the smallest positive degree in $K[x]$ such that $p(c) \in N$. By the division algorithm in $K[x]$, every generator $f(c)$ of N is easily shown to be a multiple of $p(c)$. This proves that N is the principal ideal generated by $p(c)$. $\square$

Lemma 3.6. *Let E be an arbitrary graph and K be any field. Let P be an ideal of $L_K(E)$ with $H = P \cap E^0$. Let $S = \{v \in B_H\colon\ v^H \in P\}$. Then the graded ideal $I_{(H,S)} \subset P$ contains every other graded ideal of $L_K(E)$ inside P.*

Proof. Suppose $A = I_{(H_1,S_1)} \subset P$. We claim that $A \subset I_{(H,S)}$. Clearly $H_1 \subset E^0 \cap P = H$. Note that if H_1 is empty, then clearly $A = I_{(H_1,S_1)} = \{0\} \subset I_{(H,S)}$. So we assume that H_1 is non-empty. We need to show that if $v \in S_1$ then $v^{H_1} \in I_{(H,S)}$. Thus v is a breaking vertex for H_1 and let $e_1,\dots,e_n$ be the finitely many edges satisfying $s(e_i) = v$ and $r(e_i) \notin H_1$. Suppose v is a breaking vertex for H. By re-indexing, we may then assume that for some $m \leqslant n$, $r(e_i) \notin H$ for $i = 1,\dots,m$, and that $r(e_j) \in H$ for $j = m+1,\dots,n$. Since $e_j \in I_{(H,S)}$ for $j = m+1,\dots,n$, we get $v^{H_1} = v^H - \sum_{i=m+1}^n e_i e_i^* \in I_{(H,S)}$, as desired. Suppose v is not a breaking vertex for H. Since v is a breaking vertex for H_1 but not for H, we have $r(s^{-1}(v)) \subset H$. So $e_i = e_i r(e_i) \in I_{(H,S)}$ for $i = 1,\dots,n$. As $\sum_{i=1}^n e_i e_i^* \in I_{(H,S)} \subset P$ and $v - \sum_{i=1}^n e_i e_i \in P$, we then conclude that $v \in P \cap E^0 = H$. Clearly then $v^{H_1} \in I_{(H,S)}$ and so $A \subset I_{(H,S)}$. $\square$

From Lemmas 3.3, 3.5 and 3.6, we obtain the following description of the generators of non-graded ideals in $L_K(E)$.

Corollary 3.7. *Let E be an arbitrary graph and K be any field and let N be any non-graded ideal of $L_K(E)$ with $H = N \cap E^0$ and $S = \{v \in B_H\colon\ v^H \in N\}$. Then:*

(i) *N is generated by $I_{(H,S)}$ and elements of the form $(u+\sum_{i=1}^n k_i c^{r_i})$ where c is a cycle without exits in $E^0 \backslash H$ and based at a vertex u.*
(ii) *If $L_K(E\backslash(H,S))$ is a prime ring* (*equivalently, if $E\backslash(H,S)$ satisfies the* MT-3 *condition*)*, then $E^0\backslash H$ contains exactly one cycle c without exits and N is generated by $I_{(H,S)}$ and $f(c)$ where $f(x) \in K[x]$.*

Proof. Just apply Lemmas 3.3 and 3.5 to the ideal $N/I_{(H,S)}$ of $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$. $\square$

Lemma 3.8. *Let P be a prime ideal of $L_K(E)$ with $H = P \cap E^0$ and let $S = \{v \in B_H\colon\ v^H \in P\}$. Then the ideal $I_{(H,S)}$ is also a prime ideal of $L_K(E)$.*

Proof. Suppose $A = I_{(H_1,S_1)}$ and $B = I_{(H_2,S_2)}$ are two graded ideals of $L_K(E)$ with $AB \subset I_{(H,S)}$. Since P is prime, one of them, say A, is contained in P. By Lemma 3.6, $A \subset I_{(H,S)}$. Thus the ideal $I_{(H,S)}$ is a graded prime. Since $L_K(E)$ is graded by the group $\mathbb{Z}$, as noted in the Preliminaries section, we can appeal to [20, Proposition II.1.4], to conclude that $I_{(H,S)}$ is actually a prime ideal of $L_K(E)$. $\square$

The following consequence of Lemma 3.8 may be of some interest.

Corollary 3.9. *Let E be an arbitrary graph and K be any field. Then the Leavitt path algebra $L_K(E)$ is a prime ring if and only if there is a prime ideal of $L_K(E)$ which does not contain any vertices.*

Proof. Suppose P is prime ideal of $L_K(E)$ which does not contain any vertices. If P is a graded ideal, then by [22], $P = I_{(H,S)}$ where $H = P \cap E^0$ and $S \subset B_H$. Since P contains no vertices, H and S are both empty sets and so the prime ideal $P = \{0\}$, proving that $L_K(E)$ is a prime ring. Suppose the prime ideal P is not graded and $H = P \cap E^0$. By Lemma 3.6, the ideal $I = I_{(H,B_H)}$ is then a (graded) prime ideal. Since $I \subset P$ and $I \cap E^0 = H$ by [22] and since P contains no vertices, H and hence B_H must then be empty sets. So $\{0\} = I$ is a prime ideal, showing that $L_K(E)$ is a prime ring. The converse is obvious. $\square$

The next lemma is useful in our investigation and is, perhaps, known. The proof is immediate if one uses the fact (see Proposition 10.2 of [18]) that an ideal P of a not necessarily unital ring R is a prime ideal if $P \neq R$ and whenever $aRb \subset P$ for some elements $a, b \in R$, then either $a \in P$ or $b \in P$.

Lemma 3.10. *Let R be a not necessarily unital ring and let P be a prime ideal of R. Then for any idempotent $v \in R$, vPv is a prime ideal of vRv.*

For the sake of convenience, we introduce the following definition.

Definition 3.11. A cycle c in a graph E is called a **cycle without** K, if no vertex on c is the base of another distinct cycle in E (where distinct cycles possess different sets of edges and different sets of vertices). The set of all cycles without K in the graph E is denoted by $C(E)_K$.

We are now ready to prove our main result.

In the following $\langle X, Y, a\rangle$ denotes the ideal generated by $X \cup Y \cup \{a\}$. Also recall (from Preliminaries) that for any vertex v we define $M(v) = \{w \in E^0\colon\ w \geqslant v\}$.

Theorem 3.12. *Let E be an arbitrary graph and K be any field. Let P be an ideal of $L_K(E)$ with $P \cap E^0 = H$. Then P is a prime ideal of $L_K(E)$ if and only if P satisfies one of the following conditions:*

(i) *$P = \langle H, \{v^H\colon\ v \in B_H\}\rangle$ (where B_H may be empty) and $E^0\backslash H$ satisfies the* MT-3 *condition;*
(ii) *$P = \langle H, \{v^H\colon\ v \in B_H\backslash\{u\}\}\rangle$ for some $u \in B_H$ (hence B_H is non-empty) and $E^0\backslash H = M(u)$;*
(iii) *$P = \langle H, \{v^H\colon\ v \in B_H\}, f(c)\rangle$ where c is a cycle without K in E based at a vertex u, $E^0\backslash H = M(u)$ and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.*

Proof. Now $H = P \cap E^0$. Let $S = \{w \in B_H\colon\ w^H \in P\}$.

Case 1. Suppose P is a graded ideal. Then by Theorem 5.7 [22], $P = I_{(H,S)} = \langle H, \{v^H\colon\ v \in S\}\rangle$ and that $L_K(E)/P \overset{\phi}{\cong} L_K(E\backslash(H,S))$. Thus P is a prime ideal of $L_K(E)$ if and only if $L_K(E\backslash(H,S))$ is a prime ring.

Theorem 3.1 shows that this is equivalent to $E\backslash(H,S)^0$ satisfying the MT-3 condition. Now for each $u \in B_H\backslash S$, the corresponding vertex u' is a sink in the graph $E\backslash(H,S)$. In view of the MT-3 condition, there can be at most one sink in $E\backslash(H,S)$. So $B_H\backslash S$ is either empty or a singleton $\{u\}$. Hence P is a prime ideal $L_K(E)$ if and only if either $B_H = S$ in which case $E\backslash(H,S)^0 = E^0\backslash H$ satisfies the MT-3 condition or $B_H\backslash\{u\} = S$ in which case, $(E\backslash(H,S))^0 = (E^0\backslash H) \cup \{u'\}$ and $w \geqslant u'$ for all $w \in E\backslash(H,S)^0$, equivalently, that for all $w \in E^0\backslash H$, $w \geqslant u$. Thus the primeness of the graded ideal P is equivalent to Condition (i) or (ii).

Case 2. Let P be a non-graded ideal. Suppose P is prime. By Lemma 3.8, the graded ideal $I_{(H,S)}$ is also a prime ideal of $L_K(E)$ contained in P. So, as proved in Case 1, either (i) $B_H = S$ and $E^0\backslash H$ satisfies the MT-3 condition, or (ii) $B_H\backslash\{u\} = S$ and, for all $w \in E^0\backslash H$, $w \geqslant u$. As noted earlier, $L_K(E)/I_{(H,S)} \overset{\phi}{\cong} L_K(E\backslash(H,S))$. Let $\phi(P/I_{(H,S)}) = N$. Note that, under Condition (i), $(E\backslash(H,S))^0 = (E^0\backslash H)$ and, under Condition (ii), $(E\backslash(H,S))^0 = (E^0\backslash H) \cup \{u'\}$ and $w \geqslant u'$ for all $w \in (E\backslash(H,S))^0$, where $u' = \phi(u^H + I_{(H,S)})$. Note that $u^H \notin P$, as $u \notin S$ and so $u' \notin N$. Thus, in either case, it is clear that the non-zero ideal N of $L_K(E\backslash(H,S))$ does not contain any vertices. Since $(E\backslash(H,S))^0$ satisfies the MT-3 condition, Lemma 3.5 implies that there is a cycle c based at a vertex v and without exits in the graph $E\backslash(H,S)$ such that N is the ideal generated by $f(c)$ for some polynomial $f(x) \in K[x]$. We now claim that $B_H = S$ (that is, Condition (ii) of the graded prime ideal is not possible). Because if $B_H\backslash\{u\} = S$, then $w \geqslant u'$ for all w in $(E\backslash(H,S))^0$ and since c has no exits, u' must lie on the cycle c. But this is impossible, since u' is a sink in $E\backslash(H,S)$. So $B_H = S$ must hold. Putting all these facts together, we conclude that P is the ideal generated by $H \cup \{u^H\colon\ u \in B_H\} \cup \{f(c)\}$. Now $(E\backslash(H,S))^0 = (E\backslash(H,B_H))^0 = (E^0\backslash H)$ satisfies the MT-3 condition and contains c^0. Clearly then $(E\backslash(H,S))^0 = E^0\backslash H = \{w \in E^0\colon\ w \geqslant v\} = M(v)$, where v is the base of the cycle c. It is also clear that c is a cycle without K in E. To complete the proof, we need only to show that $f(x)$ is irreducible. Now N is a prime ideal of $L_K(E\backslash(H,S))$ and hence, by Lemma 3.10, vNv is a (non-zero) prime ideal of $vL_K(E\backslash(H,S))v \overset{\theta}{\cong} K[x,x^{-1}]$ generated by $vf(c)v = f(c)$. Here the isomorphism θ maps $f(c)$ to $f(x)$ as it maps v to 1, c to x and c^* to x^{-1} (as noted in the Preliminaries). Since $f(x)$ generates the non-zero prime (hence maximal) ideal $\phi(vNv)$ in the Euclidean domain $K[x,x^{-1}]$, $f(x)$ must then be an irreducible polynomial in $K[x,x^{-1}]$.

Conversely, suppose (a) E contains a cycle c without K and based at a vertex v, (b) $E^0\backslash H = M(v)$ and (c) there exists an irreducible polynomial $f(x) \in K[x,x^{-1}]$ such that P is the ideal generated by $\{f(c)\} \cup I_{(H,B_H)}$. Now hypothesis (b) implies $(E\backslash(H,B_H))^0 = E^0\backslash H = M(v)$. So $E\backslash(H,B_H)$ satisfies the MT-3 condition and contains the cycle c. As c is a cycle without K in E, the MT-3 condition implies that c has no exits in the graph $E\backslash(H,B_H)$. Now, by Theorem 5.7 [22], $L_K(E)/I_{(H,B_H)} \overset{\phi}{\cong} L_K(E\backslash(H,B_H))$. If $N = \phi(P/I_{(H,B_H)})$, then, by hypothesis (c), the ideal N is generated by $f(c)$. As $f(x)$ is an irreducible polynomial in $K[x,x^{-1}] \overset{\theta^{-1}}{\cong} vL_K(E\backslash(H,B_H))v$, the ideal vNv, being generated by $vf(c)v = f(c) = \theta^{-1}(f(x))$, is a maximal ideal of the ring $vL_K(E\backslash(H,B_H))v$. We wish to show that N is a prime ideal of $L_K(E\backslash(H,B_H))$. Let A, B be two ideals of $L_K(E\backslash(H,B_H))$ such that $AB \subset N$. Now $vAvvBv \subset vABv \subset vNv$ implies that one of them, say $vAv \subset vNv$. We claim that A does not contain any vertices. Indeed if A contains a vertex w, then $v \in A$ as $w \geqslant v$. But then $vL_K(E\backslash(H,B_H))v \subset A$ and so $vL_K(E\backslash(H,B_H))v \subset vAv \subset vNv$, a contradiction to the fact that vNv is a proper ideal of $vL_K(E\backslash(H,S))v$. Thus A does not contain any vertices and hence, by Lemma 3.5, the ideal A of $L_K(E\backslash(H,B_H))$ will be generated by a polynomial $q(c)$. Since $q(c) = vq(c)v \in vAv \subset vNv \subset N$, we conclude that $A \subset N$. Thus we have shown that N is a prime ideal of $L_K(E\backslash(H,B_H))$. This implies that P is a prime ideal of $L_K(E)$. If P were a graded ideal, then, since $P \cap E^0 = H$, Lemma 6 implies that $P = I_{(H,B_H)}$ and this would imply that (N and hence) vNv must be 0. But $vNv \overset{\theta}{\cong} \langle f(x)\rangle \neq 0$, a contradiction. Hence P must be a non-graded prime ideal of $L_K(E)$. $\square$

As a consequence of Theorem 3.12, we get the following corollary.

Corollary 3.13. *An arbitrary graph E satisfies the Condition* (K) *if and only if every prime ideal of $L_K(E)$ is graded.*

Proof. Suppose E does not satisfy the Condition (K). Then there exists (exactly one simple closed path and hence) a cycle c without K based at a vertex v in E. Define $H = \{w \in E^0\colon\ w \ngeq v\}$. Clearly H is a hereditary saturated subset of E^0. In $E \backslash (H, B_H)$, c is then a cycle without exits and based at v. Now $vL_K(E \backslash (H, B_H))v \cong K[x, x^{-1}]$. Choose an irreducible polynomial $f(x) \in K[x, x^{-1}]$. Since $L_K(E)/I_{(H,B_H)} \overset{\phi}{\cong} L_K(E \backslash (H, B_H))$, define an ideal P containing $I_{(H,B_H)}$ such that $\phi(P/I_{(H,B_H)}) = \langle f(c) \rangle$. Then the ideal P being generated by $H \cup \{v^H\colon\ v \in B_H\} \cup \{f(c)\}$ will be a non-graded prime ideal of $L_K(E)$, by Theorem 3.12. Conversely, it is well known (see [15,22]) that Condition (K) on E implies that every ideal of $L_K(E)$ is graded. $\square$

Construction of non-graded prime ideals. The proof of Corollary 3.13 provides the following method of constructing non-graded prime ideals from a cycle without K. Suppose c is a cycle without K based at a vertex v in a graph E. Let $H = \{w \in E^0\colon\ w \ngeq v\}$. Now H is a hereditary saturated set and $E^0 \backslash H = M(v)$. So for each monic irreducible polynomial $f(x) \in K[x, x^{-1}]$, the ideal $P = \langle I_{(H,B_H)}, f(c) \rangle$ is, by Theorem 3.12, is a non-graded prime ideal of $L_K(E)$. From Theorem 3.12, it is clear that P uniquely determines and is determined by the cycle c and the polynomial $f(x)$.

Using the above construction and Theorem 3.12, we obtain the following bijective correspondence that extends Proposition 3.7 of [11] to arbitrary graphs. It is worth noting that, unlike in [11], the correspondence involves cycles without K (which are perhaps easily tractable) instead of maximal tails.

Corollary 3.14. *Let E be an arbitrary graph and K be any field. Then the assignment $P \mapsto (c, \langle f(x) \rangle)$ as indicated above and in the proof of Theorem* 3.12 *defines a bijection between non-graded prime ideals of $L_K(E)$ and the set $C(E)_\kappa \times (\mathrm{Spec}(K[x, x^{-1}]) \backslash \{0\})$ where $C(E)_\kappa$ denotes the set of all cycles without K in E* (*where cycles obtained by permuting the vertices of a cycle are considered equal*).

If we specialize Theorem 3.12 to the case of a row-finite graph E, the set B_H must be empty for any hereditary saturated set H and so Condition (ii) of Theorem 3.12 does not hold. Hence we get the following extension of Proposition 2.4 of [11] which only deals with graded prime ideals of $L_K(E)$ for row-finite graphs E.

Corollary 3.15. *Let E be a row-finite graph and K be any field. An ideal P of $L_K(E)$ with $P \cap E^0 = H$ is a prime ideal if and only if either $P = \langle H \rangle$ and $E^0 \backslash H$ satisfies the* MT-3 *condition or $P = \langle H \cup \{f(c)\} \rangle$, where c is a cycle without K in E based at a vertex v, $H = \{w \in E^0\colon\ w \ngeq v\}$ and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.*

Our investigation of prime ideals enables us to give a simpler proof the simplicity theorem (Theorem 3.11 of [1]).

Proposition 3.16. *(See [1].) Let E be an arbitrary graph and K any field. Then $L_K(E)$ is a simple ring if and only if every cycle in E has an exit and the only hereditary saturated subsets of E^0 are E^0 and the empty set Φ.*

Proof. Suppose $L_K(E)$ is a simple ring. Since it is trivially a prime ring, E^0 satisfies the MT-3 condition, by Theorem 3.1. Now the ideal generated by a non-empty proper hereditary saturated subset is a non-zero proper ideal of $L_K(E)$ and so the simplicity of $L_K(E)$ obviously implies that E^0 and the empty set are the only hereditary saturated subsets of E^0. By way of contradiction, suppose there is a cycle c without exits in E. Then since E^0 satisfies the MT-3 condition, an appeal to Theorem 3.12 yields that there are infinitely many non-graded prime ideals of $L_K(E)$ of the form $\langle f(c) \rangle$, for various irreducible polynomials $f(x)$ in $K[x, x^{-1}]$, a contradiction to the simplicity of $L_K(E)$. Thus every cycle in E has an exit.

Conversely, suppose E^0 satisfies the two conditions. Let N be a proper ideal of $L_K(E)$. Since E^0, Φ are the only hereditary saturated subsets of E^0, $N \cap E^0 = \Phi$. If N is non-zero, then by Lemma 3.3, N will be generated by elements of the form $y = (u + \sum_{i=1}^{n} k_i g^{r_i})$ where g is a cycle without exits based at a vertex u. But this contradicts the hypothesis that every cycle in E has an exit. Hence $N = 0$ and we conclude that $L_K(E)$ is simple. $\square$

Remark. In the sequel we shall be using the following observation that follows from Theorem 3.12: If $P = \langle I_{(H,B_H)}, f(c)\rangle$ and $Q = \langle I_{(H,B_H)}, g(c)\rangle$ are two non-graded prime ideals with $P \cap E^0 = H = Q \cap E^0$, then $P \nsubseteq Q$ and $Q \nsubseteq P$. This is because, if one of the proper inclusion holds, say $P \subsetneqq Q$ then, from the proof of Theorem 3.12, $f(c)$ is a divisor of $g(c)$ which will imply that in $K[x, x^{-1}]$, $f(x)$ is a proper divisor of $g(x)$, contradicting the fact that $g(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

4. Primitive ideals of $L_K(E)$

We next characterize the primitive ideals of the Leavitt path algebra $L_K(E)$. As noted in [5], the map $\phi : L_K(E) \to (L_K(E))^{op}$ given by $\sum_{i=1}^{n} k_i\alpha_i\beta_i^* \mapsto \sum_{i=1}^{n} k_i\beta_i\alpha_i^*$ is a ring isomorphism and so the distinction between the left and right primitivity of the ring $L_K(E)$ vanishes. Thus a graded ideal I of $L_K(E)$ is right primitive if and only if it is left primitive, since by [22], $L_K(E)/I$ is again a right ($=$ left) primitive Leavitt path algebra. That the same conclusion holds for non-graded ideals will follow from our internal description of primitive ideals (Theorem 4.3 below). Using Theorem 4.3, we also obtain a description of those graphs E for which every prime ideal of $L_K(E)$ is primitive.

We begin with the following important concept introduced in [5].

Definition 4.1. Let E be a graph. A subset S of E^0 is said to have the Countable Separation Property (CSP, for short), if there is a countable set C of vertices in E with the property that to each $u \in S$ there is a $v \in C$ such that $u \geqslant v$.

For example, if E is a row-finite graph, then for any vertex $v \in E$, the tree $T(v) = \{w:\ v \geqslant w\}$ is a countable set. If in addition, E^0 satisfies the MT-3 condition, then it is easy to see that E^0 will always have the CSP with respect to the tree $T(v)$ for any fixed vertex v. Also, in a countable graph E, the MT-3 condition will trivially imply the CSP for E^0.

The following characterization of primitive Leavitt path algebras of arbitrary graphs was obtained in [5].

Theorem 4.2. *(See [5].) Let E be an arbitrary graph and K be any field. Then the Leavitt path algebra $L_K(E)$ is right ($=$ left) primitive if and only if Condition* (L) *holds in E, and E^0 satisfies the* MT-3 *condition and possesses the countable separation property.*

By using Theorems 3.12 and 4.2, we are able to characterize the primitive ideals of $L_K(E)$ in the next theorem. We also need the well-known fact (see [19, Theorem 1]) that a not-necessarily unital ring R is right (left) primitive if and only if there is an idempotent $a \in R$ such that aRa is a right (left) primitive ring.

Theorem 4.3. *Let E be an arbitrary graph and K be any field. Let P be an ideal of $L_K(E)$ with $H = P \cap E^0$. Then P is right ($=$ left) primitive if and only if P satisfies one of the following*:

(i) *P is a non-graded prime ideal*;
(ii) *P is a graded prime ideal of the form $I_{(H,B_H\setminus\{u\})}$ for some $u \in B_H$*;
(iii) *P is a graded ideal of the form $I_{(H,B_H)}$ (where B_H may be empty) and $E^0\setminus H$ satisfies the* MT-3 *condition, the Condition* (L) *and the countable separation property.*

Proof. Sufficiency: (i) Suppose P is a non-graded prime ideal. We follow the notation used in the proof of the necessity part of Theorem 3.12. As noted there, the ideal $N \cong P/I_{(H,S)}$ of $L_K(E\setminus(H, S))$

is such that vNv is a maximal ideal of $vL_K(E\backslash(H,S))v \cong K[x,x^{-1}]$. Now $vL_K(E\backslash(H,S))v/vNv \cong v'(L_K(E\backslash(H,S))/N)v' \cong \bar{v}(L_K(E)/P)\bar{v}$ under the natural isomorphisms, where $v' = v + N$ and $\bar{v} = v + P$. Since $vL_K(E\backslash(H,S))v/vNv$ is a field, $\bar{v}(L_K(E)/P)\bar{v}$ is a commutative primitive ring. As noted above in the statement preceding Theorem 4.3, $L_K(E)/P$ is then a right as well as a left primitive ring, by Theorem 1 [19]. We thus conclude that P is both a right and a left primitive ideal of $L_K(E)$.

(ii) Suppose P is a graded prime ideal of the form $I_{(H,B_H\backslash\{u\})}$ for some breaking vertex u of H. By Theorem 3.12 $E^0\backslash H = M(u)$ so that $w \geqslant u$ for all $w \in E^0\backslash H$. Thus in the graph $E\backslash(H,S)$, $w \geqslant u'$ for all $w \in (E\backslash(H,S))^0$ and u' is a sink. This shows that $E\backslash(H,S)$ satisfies not only the MT-3 condition, but also the countable separation property with respect to $\{u'\}$. Moreover, Condition (L) also holds, since every vertex w on any cycle in the graph $E\backslash(H,S)$ satisfies $w \geqslant u'$ (and u' is a sink). We appeal to Theorem 4.2 to conclude that $L_K(E\backslash(H,S))$ is a right (= left) primitive ring. Since $L_K(E\backslash(H,S)) \cong L_K(E)/P$, we then conclude that P is both a right and a left primitive ideal of $L_K(E)$.

(iii) Suppose P is a graded prime ideal of the form $P = I_{(H,B_H)}$ with $E^0\backslash H$ satisfying the MT-3 condition, the Condition (L) and the countable separation property. Now by [22], $L_K(E)/P \cong L_K(E\backslash(H,B_H))$. From the definition of the graph $E\backslash(H,B_H)$, $(E\backslash(H,B_H))^0 = E^0\backslash H$ and hence satisfies the MT-3 condition, the Condition (L) and the countable separation property. By Theorem 4.2, $L_K(E\backslash(H,B_H))$ and hence $L_K(E)/P$ is then a right (= left) primitive ring. This shows that P is both a right and left primitive ideal of $L_K(E)$.

Necessity: Follows from the fact that a primitive ideal is always prime and from Theorem 4.2 and the cases (i), (ii) of Theorem 3.12. □

Theorem 4.3 establishes that the distinction between the right and the left primitivity for ideals in a Leavitt path algebra vanishes. Hence we shall drop the right/left distinction for primitive ideals of $L_K(E)$. From Theorem 4.3, we can easily describe all those graphs E for which every prime ideal of $L_K(E)$ is primitive.

Corollary 4.4. *Let E be an arbitrary graph and K any field. Then every prime ideal of the Leavitt path algebra $L_K(E)$ is primitive if and only if E satisfies the Condition* (K) *and every maximal tail M in E^0 satisfies the countable separation property.*

Proof. Suppose every prime ideal of $L_K(E)$ is primitive. First of all there cannot be any non-graded prime ideals in $L_K(E)$. Because a non-graded prime ideal P, from Theorem 3.12, is of the form $P = \langle I_{(H,B_H)}, f(c)\rangle$ and, by Lemma 3.8, it always contains the prime ideal $Q = I_{(H,B_H)}$. This leads to a contradiction, since, on the one hand, the description of P (from Theorem 3.12) implies $E^0\backslash H$ does not satisfy Condition (L) (as the cycle c has no exits in $E^0\backslash H$) and, on the other hand, the primitivity of Q implies, by Theorem 3.1, that $E^0\backslash H$ satisfies Condition (L), a contradiction. Thus every prime ideal of $L_K(E)$ is graded and hence, by Corollary 3.13, E satisfies the Condition (K). For any maximal tail M, with $H = E^0\backslash M$, if $I_{(H,S)}$ is a prime ideal (so $S = B_H$ or $B_H\backslash\{u\}$), then $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$ is a primitive ring and so, by Theorem 4.2, $(E\backslash(H,S))^0 = E^0\backslash H = M$ satisfies the Condition (L) and the countable separation property.

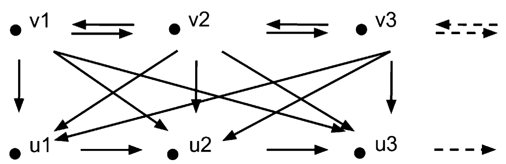
Conversely, it is easy to see that the Condition (K) in E implies that any maximal tail in E^0 also satisfies the Condition (K) and hence the Condition (L). Moreover, every prime ideal of $L_K(E)$ is graded. For any graded prime ideal $I_{(H,S)}$, $(E\backslash(H,S))^0 = E^0\backslash H$ is a maximal tail that satisfies, by hypothesis, the Condition (L) and the countable separation property. So $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$ is a primitive ring by Theorem 4.2. This implies that $I_{(H,S)}$ is a primitive ideal. □

Here are a few graphs E satisfying the conditions of Corollary 4.4 (and thus every prime ideal of the corresponding $L_K(E)$ is primitive): (a) E is any countable acyclic graph; (b) E is a row-finite graph satisfying the Condition (L); (c) E is a graph consisting of a straight line graph $\overset{v_1}{\bullet} \to \overset{v_2}{\bullet} \to \overset{v_3}{\bullet} \to \bullet \to \bullet \cdots$ together with a cycle $u \leftrightarrows w$ where, for each n, there is an edge from u to v_n. Thus u is a breaking vertex for $H = \{v_1, v_2, \ldots\}$. Here $M = \{u, w\}$ is the only maximal tail in E and it satisfies Condition (i) of Corollary 4.4.

4.1. Examples

We now construct various examples of prime and primitive ideals of Leavitt path algebras illustrating the various possibilities mentioned in Theorems 3.12 and 4.3.

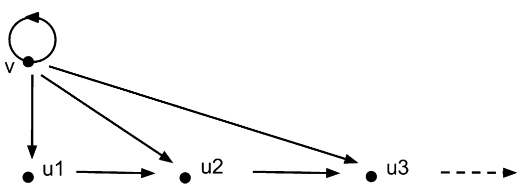
Example 4.5. Consider the graph E where $E^0 = \{v_i, u_i\colon\ i = 1, 2, 3, \ldots\}$. For each i, there is exactly one edge from u_i to u_{i+1}. For each j, each v_j is an infinite emitter connected to each u_i by an edge. Also each v_j connects to v_{j+1} by an edge and vice versa (to form a cycle of length 2). Thus E looks like the graph below.



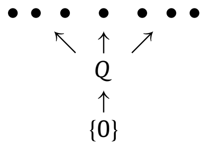
It is easy to check that there are only three hereditary saturated subsets of vertices in the graph E, namely, $H = \{u_1, u_2, \ldots\}$, E^0 and the empty set. Now $B_H = \{v_1, v_2, \ldots\}$. For each $j = 1, 2, \ldots,$ let $S_j = B_H \backslash \{v_j\}$. Observe that in $E^0\backslash H$, $v_i \geqslant v_j$ and $v_j \geqslant v_i$ for any two i and j. It is then clear that E^0, $E^0\backslash H$ and, for each j, $(E\backslash(H, S_j))^0 = (E^0\backslash H) \cup \{v'_j\}$ all satisfy the MT-3 condition. As an application of Theorem 3.12, we then get, for each j, the ideal $P_j = I_{(H,S_j)}$, the ideal $P = I_{(H,B_H)}$ and $\{0\}$ are all graded prime ideals of $L_K(E)$. Note that E satisfies the Condition (K). Further E^0, being countable, has the countable separation property. We thus conclude the following: (i) $\{0\}$, P, P_j $(j \geqslant 1)$ are the only prime ideals of $L_K(E)$; (ii) every prime ideal of $L_K(E)$ is graded and (iii) all the prime ideals of $L_K(E)$ are primitive (by Corollary 4.4).

Remark. In Example 4.5, every prime ideal of $L_K(E)$ is graded. A natural question is: Is there a graph E such that every prime ideal in $L_K(E)$ is non-graded? The answer is in the negative. Indeed, as shown in Lemma 3.8 and in view of Theorem 3.12, if J is a non-graded prime ideal of $L_K(E)$ with $H = J \cap E^0$, then the ideal $I_{(H,B_H)}$ is always a graded prime ideal of $L_K(E)$.

Example 4.6. Consider the graph F given below.

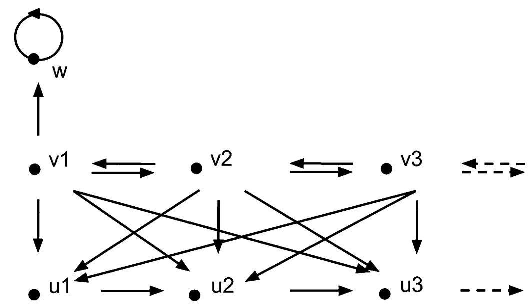


Here v is an infinite emitter, connected to every vertex u_j. Let c denote the loop at v. Now $H = \{u_1, u_2, \ldots\}$ is the only proper non-empty hereditary saturated subset of E^0 with $B_H = \{v\}$ and c has no exits in $E^0\backslash H$. Let J be the two-sided ideal generated by $H \cup \{v - cc^*\} \cup \{v - c\}$. By Theorem 3.12, J is a non-graded prime ideal of $L_K(F)$ and is thus primitive. Note that the non-graded prime (= primitive) ideal of $L_K(F)$ are in one-to-one correspondence with the irreducible polynomials in $K[x.x^{-1}]$. By Theorem 3.12, the ideal $Q = I_{(H,B_H)} = \langle H, v - cc^*\rangle$ is a graded prime ideal of $L_K(E)$ and is not primitive as $E^0\backslash H$ does not satisfy Condition (L). Also $\{0\}$ is a (graded) prime (actually primitive) ideal of $L_K(E)$ as E^0 satisfies the MT-3 condition, the Condition (L) and the CSP. The poset *Spec*$(L_K(F))$ under set inclusion can be pictorially described as



where there are infinitely many dots • denoting the infinitely many non-graded prime ideals of $L_K(F)$.

Example 4.7. Let G be the graph given below.



This graph G is essentially the graph E of Example 4.5 except for an additional vertex w, an edge from v_1 to w and a loop at w denoted by c. In addition to G^0 and the empty set, there are two hereditary saturated subsets in G^0, namely, $H_1 = \{u_1, u_2, \ldots\}$ and $H_2 = \{u_1, u_2, \ldots\} \cup \{w\}$. Note that $B_{H_1} = B_{H_2} = \{v_1, v_2, \ldots\}$. $L_K(G)$ has infinitely many graded prime ideals and also infinitely many non-graded prime ideals. For example, $G^0 \backslash H_1 = M(w)$ satisfies the MT-3 condition and has a cycle/loop c without exits based at w. For each irreducible polynomial $f(x) \in K[x, x^{-1}]$, the ideal $P_{f(x)}$ generated by $H_1 \cup \{v^{H_1}\colon\ v \in B_{H_1}\} \cup \{f(c)\}$ is, by Theorem 3.12, a non-graded prime ideal of $L_K(G)$. Since $G^0 \backslash H_2 = M(v_i)$ for each v_i, the ideal $I_{(H_2, S_i)}$ is a graded prime ideal of $L_K(G)$ where $S_i = B_{H_2} \backslash \{v_i\}$. Note that the graded ideals $I_{(H_2, S_i)}$ are all primitive. On the other hand, the graded ideal $I_{(H_1, B_{H_1})}$ is a prime ideal which is not primitive.

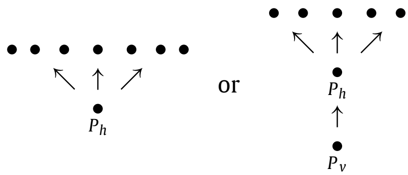
5. The stratification of *Spec*$(L_K(E))$

In the case of an arbitrary graph E, the stratification of the prime spectrum of $L_K(E)$ is somewhat similar, but different from the stratification of $Spec(L_K(E))$ for a row-finite graph E [4]. For each cycle c without K and based at a vertex v in the arbitrary graph E, let $M_c = \{w \in E^0\colon\ w \geqslant v\}$ and $H_c = \{w \in E^0\colon\ w \ngeqslant v\}$. Note that M_c is the smallest maximal tail containing c. For each maximal tail M in E^0, define the *stratum corresponding to* M with $E^0 \backslash M = H$ to be

$$Spec_M\big(L_K(E)\big) = \big\{P \in Spec\big(L_K(E)\big)\colon\ P \cap E^0 = H\big\}.$$

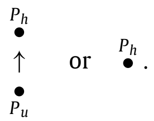
Thus the prime spectrum of the Leavitt path algebra $L_K(E)$ is the union of disjoint strata corresponding to distinct maximal tails.

For a given cycle c without K and based at a vertex v with $M_c = \{w \in E^0\colon\ w \geqslant v\}$, it is clear from the proof of Theorem 3.12 that, the corresponding stratum $Spec_{M_c}(L_K(E))$ consists of (i) at most two graded prime ideals $P_h = I_{(H, B_H)}$ and, in case $v \in B_H$, $P_v = I_{(H, B_H \backslash \{v\})}$, where $H = \{w \in E^0\colon\ w \ngeqslant v\}$ and (ii) an infinite set of non-graded prime ideals all containing the ideal $I_{(H, B_H)}$ and indexed by the irreducible polynomials in $K[x, x^{-1}]$. Thus if $v \notin B_H$, P_h will be the only graded prime ideal of $Spec_{M_c}(L_K(E))$ and in this case, $Spec_{M_c}(L_K(E))$ is homeomorphic to $Spec(K[x, x^{-1}])$ with P_h corresponding to the ideal $\{0\}$ of $K[x, x^{-1}]$. The poset of $Spec_{M_c}(L_K(E))$ under set inclusion will look like



where the dots $\bullet$ denote infinitely many non-graded prime ideals of $L_K(E)$. Observe that in the above case, the cycle c has no exits in M_c.

If M is a maximal tail in which every cycle has an exit, it determines at most two graded prime ideals $P_u = I_{(H,B_H\setminus\{u\})}$ (when there is a $u \in B_H$ such that $H = \{w \in E^0\colon\ w \ngeqslant u\}$) and $P_h = I_{(H,B_H)}$ (with P_h being the only graded prime ideal, if $B_H = \Phi$ or if $E^0\setminus H \neq M(u)$ for any $u \in B_H$). Note that $P_u \subset P_h$ and that P_u will always be primitive by Theorem 4.3. Thus, in this case, the poset of $Spec_M(L_K(E))$ under set inclusion will look like



6. Leavitt path algebras with prescribed Krull dimension

Recall that a ring R is said to have **Krull dimension** n, if n is the supremum of all non-negative integers k such that there is a chain of prime ideals $P_0 \subsetneqq \cdots \subsetneqq P_k$. We can extend this definition to infinite ordinals λ. Thus R is said to have Krull dimension λ, if λ is the supremum of the order types of all the continuous well-ordered of ascending chains of prime ideals in R.

In this section, we characterize the Leavitt path algebras with Krull dimension 0. We also construct examples of Leavitt path algebras with various prescribed Krull dimensions (both finite and infinite).

We begin by describing the graphical conditions on E under which every non-zero prime ideal of $L_K(E)$ is maximal.

Recall that every non-empty subset X of vertices in a graph E gives rise to the restricted subgraph E_X where $(E_X)^0 = X$ and $(E_X)^1 = \{e \in E^1\colon\ s(e), r(e) \in X\}$.

Theorem 6.1. *Let E be an arbitrary graph and K be any field. Then every non-zero prime ideal of the Leavitt path algebra $L_K(E)$ is maximal if and only if E satisfies one of the following two conditions*:

Condition I: (i) *E^0 is a maximal tail*; (ii) *The only hereditary saturated subsets of E^0 are E^0 and Φ* (*the empty set*); (iii) *E does not satisfy the Condition* (K).

Condition II: (a) *E satisfies the Condition* (K); (b) *For each maximal tail M, the restricted graph E_M contains no proper non-empty hereditary saturated subsets*; (c) *If H is a hereditary saturated subset of E^0, then for each $u \in B_H$, $M(u) \subsetneqq E^0\setminus M$.*

Proof. Suppose every non-zero prime ideal of $L_K(E)$ is a maximal ideal. We distinguish two cases.

Case 1. Suppose there exists a non-graded prime ideal P in $L_K(E)$. We shall show that E satisfies Condition I. From Theorem 3.12, $P = \langle I_{(H,B_H)}, f(c)\rangle$, where $H = P \cap E^0$ and c is a cycle without K based at a vertex v in E and that $E^0\setminus H = M(v)$. Since, by Lemma 3.8, P contains the graded prime ideal $I_{(H,B_H)}$ which, if non-zero, must be a maximal ideal, we conclude that $H = \Phi$. So $P = \langle f(c)\rangle$ and contains no vertices. Now $E^0\setminus H = E^0$ and $w \geqslant v$ for every $w \in E^0$. The last property implies (as c is a cycle without K) that c has no exits in E^0 (thus, in particular, E does not satisfy the Condition (K)) and that E^0 is a maximal tail. It also implies that any non-empty hereditary saturated subset X of E^0 contains c^0. So if $Q = \langle X\rangle$, then Q contains c and hence properly contains $\langle f(c)\rangle = P$. By the maximality of P, $Q = L_K(E)$ and so $X = E^0$. Thus, Φ and E^0 are the only hereditary saturated subsets of E^0. (In particular, all the non-zero ideals of $L_K(E)$ are non-graded.) This proves that E satisfies Condition I. (Note also that, in this case, $L_K(E)$ is a prime ring, by Theorem 3.1.)

Case 2. Suppose every prime ideal of $L_K(E)$ is graded. So, by Corollary 3.13, the graph E satisfies the Condition (K). We wish to establish Condition II (b), (c) for E. Now Condition II (c) must hold, since otherwise there will be a hereditary saturated subset H of E^0 and a $u \in B_H$ such that $M(u) = E^0 \backslash H$. Then, by Theorem 3.12, the ideal $I_{(H, B_H \backslash \{u\})}$ will be a graded prime ideal and, since it is properly contained in $I_{(H,B_H)}$, we get a contradiction to the maximality of $I_{(H, B_H \backslash \{u\})}$. To prove Condition II (b), let M be any maximal tail with $E^0 \backslash M = H$. By Theorem 3.12 (i), $I_{(H,B_H)}$ is a prime ideal and hence by supposition, a maximal ideal of $L_K(E)$. Then $L_K(E\backslash(H, B_H)) \cong L_K(E)/I_{(H,B_H)}$ is a simple ring and so, by Theorem 3.11 of [1], $M = E^0 \backslash H = (E\backslash(H, S))^0$ contains no proper non-empty hereditary saturated subsets. Thus E satisfies Condition II.

Conversely, suppose E satisfies Condition I of the theorem. Since Condition (K) does not hold, $L_K(E)$ cannot be a simple ring by Lemma 4.1 of [2] and so there are non-zero ideals in $L_K(E)$. By hypothesis I (ii) all the non-zero ideals of $L_K(E)$ are non-graded. Moreover, hypotheses I (i), (ii), together with Lemma 3.5, imply that there exists a cycle c without exits in E^0 and that every non-zero ideal J of $L_K(E)$ is of the form $\langle g(c) \rangle$ for some polynomial in $g(x) \in K[x] \subset K[x, x^{-1}]$. Let P be a non-zero prime ideal of $L_K(E)$. Since P is non-graded, by Theorem 3.12, $P = \langle p(c) \rangle$ where $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. If $J = \langle g(c) \rangle$ is an ideal with $J \not\subseteq P$, then $g(x) \notin \langle p(x) \rangle$ and by maximality, $\langle g(x) \rangle + \langle p(x) \rangle = K[x, x^{-1}]$. Since $vL_K(E)v \cong K[x, x^{-1}]$, we conclude that $v \in vJv + vPv \subset J + P$. Thus $(J + P) \cap E^0 \neq \Phi$ and so, by hypothesis I (ii), $E^0 \subset J + P$. Hence $J + P = L_K(E)$. This proves that P is a maximal ideal of $L_K(E)$.

Suppose now E satisfies the Condition II. Now the Condition (K) implies that every ideal of $L_K(E)$ is graded. By Condition II (c) and Theorem 3.12 (i), (ii), every non-zero prime ideal P of $L_K(E)$ is of the form $I_{(H,B_H)}$ where $H = P \cap E^0$. Let $M = E^0 \backslash H$. Now the Condition (K) in E implies that $E\backslash(H, B_H)$ satisfies the Condition (K) and, since $(E\backslash(H, B_H))^0 = E^0 \backslash H = M$, our hypothesis implies that $E\backslash(H, B_H) = E_M$ contains no non-empty proper hereditary saturated subsets of vertices and so, by Theorem 3.11 of [1], $L_K(E)/I_{(H,B_H)} \cong L_K(E\backslash(H, B_H))$ is a simple ring, thus proving that $I_{(H,B_H)}$ is a maximal ideal of $L_K(E)$. $\square$

When every non-zero prime ideal of $L_K(E)$ is maximal and $L_K(E)$ contains a non-graded prime ideal P, then our proof of Case I in Theorem 6.1 shows that P contains no vertices and there is a unique cycle c based at a vertex v and that $w \geqslant v$ for every $w \in E^0$. Gene Abrams points out that if further, E is a finite graph, then in this case, $L_K(E) \cong M_n(K[x, x^{-1}])$ where n is the number of paths in E which end in c but do not contain c. This was shown in Theorem 3.3 of [3]. On the other hand, since $K[x, x^{-1}]$ is a Euclidean domain, every non-zero prime ideal of $K[x, x^{-1}]$ and hence also of $M_n(K[x, x^{-1}])$ is maximal. Thus we get the following corollary.

Corollary 6.2. *Let E be a finite graph. Then every non-zero prime ideal of $L_K(E)$ is maximal if and only if either $L_K(E) \cong M_n(K[x, x^{-1}])$ for some positive integer n or E satisfies the Condition (K) and, for each maximal tail M, the restricted graph E_M contains no proper non-empty hereditary saturated subsets of vertices.*

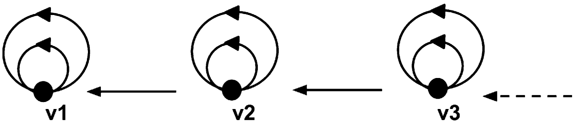
Next we shall use Theorem 6.1 and Proposition 3.16 to describe those Leavitt path algebras whose Krull dimension is zero.

Recall that a subset S of a partially ordered set $(X, \leqslant)$ is called an **antichain** if for any two $a, b \in S$, we have $a \nleqslant b$ and $b \nleqslant a$.

We begin with some preliminary observations. Clearly, a maximal ideal M of a ring R with multiplicative identity 1 is a prime ideal of R. If R is a ring without identity but $R^2 = R$ (such as a Leavitt path algebra), then also a maximal ideal M of R is prime. This is because, if A, B are two ideals of R such that $A \not\subseteq M$ and $B \not\subseteq M$ so that $A + M = R$ and $B + M = R$, then $AB \not\subseteq M$ since otherwise $R = R^2 = (A + M)(B + M) = AB + AM + MB + MM \subset M$, a contradiction.

If R is a ring with identity, then, by Zorn's Lemma, every (prime) ideal of R embeds in a maximal ideal and so the Krull dimension of R is 0 if and only if every prime ideal of R is maximal. But this no longer is true in a Leavitt path algebra $L_K(E)$ if E is an arbitrary graph. Because, as the following example shows, maximal ideals need not even exist in $L_K(E)$. (In contrast, it is worth noting that maximal one sided ideals do always exist in any $L_K(E)$: Select a vertex v in E and apply Zorn's Lemma to pick a maximal left $L_K(E)$-submodule M of $L_K(E)v$. Then $M \oplus C$ is a desired maximal left ideal of $L_K(E)$, where $L_K(E) = L_K(E)v \oplus C$.)

Example. Let E be the graph consisting of an infinite line segment in which there are two loops at each vertex v_i and, for each i, there is an edge from vertex v_{i+1} to vertex v_i. Thus E looks like:



Since E satisfies the Condition (K), every ideal in $L_K(E)$ is graded and is generated by vertices (since E is row-finite). Now the proper non-empty hereditary saturated subsets of E^0 are precisely the sets $H_n = \{v_1, \dots, v_n\}$ for various positive integers n. So the non-zero ideals of $L_K(E)$ are of the form $\langle H_n \rangle$, all of which are actually prime ideals of $L_K(E)$ (since $E^0 \backslash H_n$ is a maximal tail for each n). These ideals form an ascending chain $\{0\} \subset H_1 > \subset \cdots \subset \langle H_n \rangle \subset \cdots$. Obviously $L_K(E)$ has no maximal ideals.

Theorem 6.3. *Let E be an arbitrary graph and K be any field. Then the Leavitt path algebra $L_K(E)$ has Krull dimension* 0 *if and only if E satisfies the Condition* (K)*, the maximal tails in E^0 form an antichain under set inclusion and no maximal tail M is of the form $M = M(u)$ for any $u \in B_H$ where $H = E^0 \backslash M$.*

Proof. Suppose $L_K(E)$ has Krull dimension 0. This means that if P is a prime ideal of $L_K(E)$, then there cannot be another prime ideal Q with $P \subsetneqq Q \subsetneqq L_K(E)$.

We claim that there cannot be a non-graded prime ideal P in $L_K(E)$. Because, by Theorem 3.12, such a P (with $H = P \cap E^0$) will be of the form $P = \langle I_{(H,B_H)}, f(c) \rangle$. Since $P \supsetneqq I_{(H,B_H)}$ which, by Lemma 3.8, is also a prime ideal, we get a contradiction to the fact that $L_K(E)$ has Krull dimension 0. Thus every prime ideal of $L_K(E)$ must be graded and so, by Corollary 3.13, E satisfies the Condition (K).

Suppose $\{0\}$ is a prime ideal of $L_K(E)$. Then, by Theorem 3.1, E^0 is a maximal tail and $L_K(E)$ has no other prime ideals. So there are no non-empty proper maximal tails. So trivially the needed conditions on maximal tails are satisfied.

Suppose $\{0\}$ is not a prime ideal so E^0 is not a maximal tail. Let M_1, M_2 be two distinct maximal tails in E^0. If one is contained in the other, say, $M_1 \subseteq M_2$ with $H_1 = E^0 \backslash M_1$ and $H_2 = E^0 \backslash M_2$, then we get the prime ideals $I_{(H_2,B_{H_2})} \subsetneqq I_{(H_1 B_{H_1})} \neq L_K(E)$, contradicting the fact that $L_K(E)$ has Krull dimension 0. Thus the maximal tails in E^0 form an antichain under set inclusion. If a maximal tail $M = M(u)$ for some $u \in B_H$ where $H = E^0 \backslash M$, then, by Theorem 3.12, we get the prime ideals $I_{(H,B_H)} \supsetneqq I_{(H,B_H \backslash \{u\})}$, again contradicting that $L_K(E)$ has Krull dimension 0. This proves the necessity.

To prove the sufficiency, observe that, Condition (K) and Theorem 3.12 imply that every prime ideal in $L_K(E)$ must be of the form $I_{(H,S)}$ with $E^0 \backslash H = M$ a maximal tail, where $S = B_H$ or $B_H \backslash \{u\}$ for some $u \in B_H$. Note the latter case is not possible, since, in that case, by Theorem 3.12, $M = M(u)$, contradicting the hypothesis. Thus all the prime ideals of $L_K(E)$ must be of the form $I_{(H,B_H)}$. Let $P_1 = I_{(H_1,B_{H_1})}$ be a prime ideal of $L_K(E)$. If there is another prime ideal $Q = I_{(H_2,B_{H_2})}$ such that $P \subsetneqq Q$, then we get maximal tails $E^0 \backslash H_2 \subsetneqq E^0 \backslash H_1$, contradicting the fact that the maximal tails in E^0 form an antichain under set inclusion. Thus $L_K(E)$ has Krull dimension 0. $\square$

When E is a finite graph, Theorem 6.3 reduces to the following corollary.

Corollary 6.4. *Let E be a finite graph. Then the Leavitt path algebra $L_K(E)$ has Krull dimension zero if and only if E satisfies the Condition* (K) *and for every maximal tail M in E^0, the restricted graph E_M contains no non-empty proper hereditary saturated subset of vertices.*

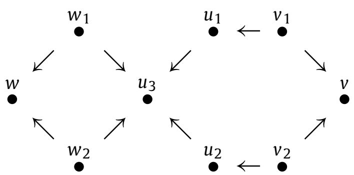
Proof. Suppose $L_K(E)$ has Krull dimension zero. By Theorem 6.3, E satisfies the Condition (K) and, in particular, every prime ideal I of $L_K(E)$ is a graded ideal. Since the graph E is finite, we get from Theorem 3.12 that I of the form $I = \langle H \rangle$, where $H = I \cap E^0$ with $M = E^0 \backslash H$ a maximal tail. Moreover, I is a maximal ideal of $L_K(E)$. Then $L_K(E \backslash (H)) \cong L_K(E)/I$ is a simple ring and Proposition 3.16 implies that E_M, where $M = (E \backslash H)^0$, contains no proper non-empty hereditary saturated subsets.

Conversely, Condition (K) implies that every ideal of $L_K(E)$ is graded and so every prime ideal P of $L_K(E)$ is of the form $P = \langle H \rangle$ where $H = P \cap E^0$ with $M = E^0 \backslash H$ a maximal tail. By hypothesis, $(E \backslash H)^0 = (E_M)^0$ has no proper non-empty hereditary saturated subsets and $E \backslash H$ satisfies the Condition (K). So, by Proposition 3.16, $L_K(E)/P \cong L_K(E \backslash (H))$ is a simple ring showing that P is a maximal ideal of $L_K(E)$. Hence $L_K(E)$ has Krull dimension zero. □

6.1. Examples

The following examples illustrate graphs satisfying the properties stated in Theorems 6.1 and 6.3, and Corollary 6.4.

Example 6.5. Let E_1 be the graph.

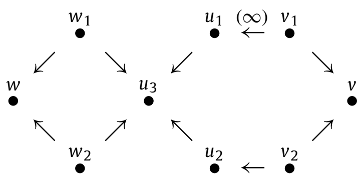


The graph E_1 is acyclic (hence trivially satisfies Condition (K)), contains three non-trivial hereditary saturated subsets

$$H = \{u_1, u_2, u_3\}, \qquad H_1 = \{w, w_1, w_2, u_1, u_2, u_3\}, \qquad H_2 = \{u_1, u_2, u_3, v_1, v_2, v\}$$

and two maximal tails $M_1 = E_1^0 \backslash H_1 = \{v_1, v_2, v\}$ and $M_2 = E_1^0 \backslash H_2 = \{w, w_1, w_2\}$. Clearly, M_1 and M_2 contain no non-empty proper hereditary saturated subsets of vertices. Thus E_1 satisfies the second condition of Corollary 6.4. The prime (= maximal) ideals of $L_K(E_1)$ are $\langle H_1 \rangle$ and $\langle H_2 \rangle$. Thus $L_K(E_1)$ has Krull dimension 0. Note that $\langle H \rangle$ is not a prime ideal as $E_1^0 \backslash H$ is not a maximal tail.

Example 6.6. Let E_2 be the graph.



Now E_2 is the same as the graph E_1 with the exception that v_1 is now an infinite emitter with $r(s^{-1}(v_1)) = \{u_1\}$ as indicated by $\overset{u_1}{\bullet} \overset{(\infty)}{\leftleftarrows} \overset{v_1}{\bullet}$. Here $B_{H_1} = \{v_1\}$. Since $v \not\geq v_1$, Condition II (c) of Theorem 6.1 is satisfied. As before the Conditions II (a), (b) are also satisfied by E_2. The prime ideals of $L_K(E_2)$ are $I_{(H_1, B_{H_1})}$ and $I_{(H_2, \Phi)}$. Thus $L_K(E_2)$ has Krull dimension 0.

Example 6.7. Let E_3 be the graph

$$\begin{tikzcd}
& w_1 \arrow[dl] \arrow[dr] & & u_1 \arrow[dl] & v_1 \arrow[l, "(\infty)"'] \arrow[dr, shift left] & \\
w & & u_3 & & & v \arrow[ul, shift left] \\
& w_2 \arrow[ul] \arrow[ur] & & u_2 \arrow[ul] & v_2 \arrow[l] \arrow[ur] &
\end{tikzcd}$$

The graph E_3 is obtained from E_2 by adding an edge connecting the vertex v to v_1. Now the ideals $I_{(H_1,B_{H_1})}$ and $I_{(H_1,B_{H_1\setminus\{v_1\}})}$ are both prime ideals, but $I_{(H_1,B_{H_1\setminus\{v_1\}})}$ is not a maximal ideal since $I_{(H_1,B_{H_1\setminus\{v_1\}})} \subsetneqq I_{(H_1,B_{H_1})}$. Thus the Krull dimension of $L_K(E_3)$ is not 0. Note that Condition II (c) of Theorem 6.1 is not satisfied.

Example 6.8.

(a) Let E_4 be the graph.

$$\begin{tikzcd}
& v_4 \arrow[d] & v_3 \arrow[l] \\
u \arrow[r, shift left] & v_1 \arrow[l, shift left] \arrow[r] & v_2 \arrow[u]
\end{tikzcd}$$

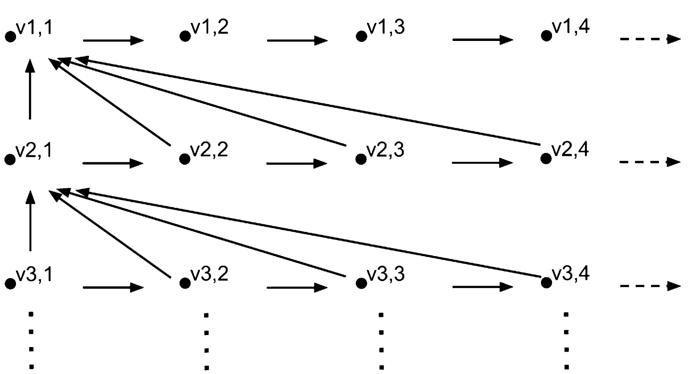
Clearly E_4 satisfies Condition (L), the MT-3 condition and has no non-empty proper hereditary saturated subset of vertices. Hence $L_K(E_4)$ is a prime simple ring and has Krull dimension 0. The same conclusion holds if in the graph E_5 we replace the cycle $v_1v_2v_3v_4$ by an arbitrary cycle c.

(b) Let E_5 be the graph with a single vertex and a single loop. Clearly E_5 satisfies the Condition I of Theorem 6.1 and $L_K(E_5) \cong K[x, x^{-1}]$, which being a Euclidean domain has all its non-zero prime ideals maximal. Since $K[x, x^{-1}]$ is an integral domain, $\{0\}$ is a prime ideal and $L_K(E_5)$ has Krull dimension one.

We now construct Leavitt path algebras of any prescribed Krull dimension.

Example 6.9. For each $n \leqslant \omega$, there exists a Leavitt path algebra $L_K(E)$ with Krull dimension n.

Let $P_\omega = \bigcup_{n\in\mathbb{N}} P_n$ be the "Pyramid" graph of length ω constructed inductively in [7] and represented pictorially as follows.



Specifically, the graph P_1 consists of the infinite line segment in the first row of P_ω. The pyramid graph P_2 consists of the vertices in the top two infinite line segments together with all the edges they emit in P_ω. More generally, for any $n \geqslant 1$, the pyramid graph P_n consists of all the vertices in the first n "rows" of P_ω together with all the edges they emanate.

(i) **Claim.** *For each integer n, the Krull dimension of $L_K(P_{n+1})$ is n.*
Now the graph $E = P_{n+1}$ is row-finite and contains the chain of pyramid subgraphs $P_1 \subsetneqq \cdots \subsetneqq P_n$ where P_i is embedded in P_{i+1} by identifying P_i with the top i "layers" of P_{i+1}. Observe that for each $i = 1, \dots, n$, $(P_i)^0$ is a hereditary saturated subset of E^0 and $E^0 \backslash (P_i)^0$ satisfies the MT-3 condition and so, the ideal $J_i = \langle (P_i)^0 \rangle$ is a prime ideal of $L_K(E)$. Also E^0 satisfies MT-3 and so, by Theorem 3.1, $\{0\}$ is a prime ideal. Thus we get a chain of prime ideals $J_0 = \{0\} \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_n$. Moreover, $(P_1)^0, \dots, (P_n)^0$ are the only proper non-empty hereditary saturated subsets of E^0 and so $J_1, \dots, J_n$ are the only non-zero ideals of $L_K(E)$. This proves that the Krull dimension of $L_K(P_{n+1})$ is exactly n.
(ii) Since $P_\omega = \bigcup_{n<\omega} P_n$, it is then clear that $L_K(P_\omega)$ is the union of the ascending chain of prime ideals $J_0 = \{0\} \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_n \subsetneqq \cdots$ where $J_n = \langle (P_n)^0 \rangle$. Thus $L_K(P_\omega)$ has Krull dimension ω.
(iii) Similarly, using the transfinite construction of the pyramid graphs P_κ for various infinite ordinals κ as given in [7], we can establish the existence of Leavitt path algebras having Krull dimension κ for various infinite ordinals κ.

Properties of $L_K(P_\kappa)$. The Leavitt path algebra $L_K(P_\kappa)$ of the pyramid graph P_κ has many interesting properties: (i) It is a primitive von Neumann regular ring (as E is acyclic; see [8]); (ii) Every ideal of $L_K(P_\kappa)$ is a primitive ideal and is further graded (justification: E is acyclic, E^0 satisfies the MT-3 condition and every hereditary saturated subset of E^0 satisfies the CSP with respect to the countable set of vertices in the "first layer", namely, the set $\{v_{11}, v_{12}, v_{13}, \dots\}$); (iii) $L_K(P_\kappa)$ is "two-sided" uniserial (that is, its ideals form a chain); (iv) Moreover, the chain $J_0 = \{0\} \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_i \subsetneqq \cdots$ where $J_i = \langle (P_i)^0 \rangle$ is a "two-sided composition series" or "saturated" in the sense that, for each i, J_i is an ideal with J_{i+1}/J_i a simple ring; (v) The ideals chain of $L_K(P_\kappa)$ is the socle series for $L_K(P_\kappa)$, that is, $J_{i+1}/J_i = soc(L_K(P_\kappa)/J_i)$, for all $i \geqslant 0$.

7. Minimal prime ideals of $L_K(E)$

In this section, we characterize those ideals of an arbitrary Leavitt path algebra which are minimal and also construct examples of such ideals. Recall that a prime ideal P of a ring R is said to be a *minimal prime ideal* if there is no prime ideal Q satisfying $Q \subsetneqq P$. Recall that for any vertex v in a graph E, the tree of v is the set $T(v) = \{w \in E^0\colon\ v \geqslant w\}$ and $M(v) = \{w \in E^0\colon\ w \geqslant v\}$.

For convenience in expression, we introduce the following definition.

Definition 7.1. A hereditary saturated subset H of vertices in the graph E is said to have the Property $(*)$, if every non-empty proper maximal tail S in H contains a vertex u such that $T(u) \cap T(v) = \Phi$ (the empty set) for some $v \notin H$.

Theorem 7.2. *Let E be an arbitrary graph and K be any field. Then a non-zero prime ideal P of $L_K(E)$ with $P \cap E^0 = H$ is a minimal prime ideal if and only if P is a graded prime ideal such that H satisfies the Property $(*)$ and either $P = I_{(H, B_{H \backslash \{v\}})}$ for some $v \in B_H$ or $P = I_{(H, B_H)}$ with $E^0 \backslash H \neq M(v)$ for all $v \in B_H$.*

Proof. Let P be a non-zero minimal prime ideal. Now P cannot be a non-graded prime ideal, since by Theorem 3.12 (iii), P is then of the form $P = \langle I_{(H, B_H)}, p(c) \rangle$ and by Lemma 3.8, $I_{(H, B_H)} \subsetneqq P$ will (even if 0) always be a prime ideal. Thus P must be a graded ideal of $L_K(E)$. If H does not have the Property $(*)$, then there is a maximal tail S in H such that for all $u \in S$ and $v \in E^0 \backslash H$, $T(u) \cap T(v) \neq \Phi$ and so there is a $w \in T(u) \cap T(v)$ such that $u \geqslant w$ and $v \geqslant w$. It is then readily seen that $M = S \cup (E^0 \backslash H)$ is a maximal tail in E^0. Then the ideal generated by $E^0 \backslash M = H \backslash S$ is, by Theorem 3.12, a non-zero prime ideal of $L_K(E)$ contained in P, thus contradicting the minimality of P. Also if P is of the form $I_{(H,B_H)}$, and $M(v)=E^0\backslash H$ for some $v\in B_H$, then $I_{(H,B_H)}$ will properly contain the prime ideal $I_{(H,B_{H\backslash\{v\}})}$, a contradiction to the minimality of $I_{(H,B_H)}$.

Conversely, let $P\neq 0$ be a graded prime ideal of $L_K(E)$ satisfying the given hypotheses. By Theorem 3.12, P is of the form $I_{(H,B_{H\backslash\{u\}})}$ for some $u\in B_H$ or of the form $I_{(H,B_H)}$. We claim that there is no prime ideal $J\subsetneqq P$ such that $J\cap E^0=H$. This is clear (by Theorem 3.12) when P is of the form $I_{(H,B_{H\backslash\{u\}})}$. On the other hand, if P is of the form $I_{(H,B_H)}$ and if there exists such a prime ideal J with $J\cap E^0=H$, then J must be of the form $I_{(H,B_{H\backslash\{v\}})}$ for some $v\in B_H$ in which case $M(v)=E^0\backslash H$, a contradiction to the hypothesis. Suppose now that there is a non-zero prime ideal $J\subsetneqq P$ with $J\cap E^0=X\subsetneqq H$. By Theorem 3.12, $M=E^0\backslash X$ is a maximal tail in E^0. Clearly $S=H\cap M$ is a maximal tail in H. But this S contradicts the Property $(*)$ of H, since, by the MT-3 condition of M, for any $u\in S\subset M$ and $v\in E^0\backslash H\subset M$ there is a $w\in M$ such that $u\geqslant w$, $v\geqslant w$ showing that $T(u)\cap T(v)\neq\Phi$. This proves that P is a minimal prime ideal of $L_K(E)$. $\square$

Example. Consider the graph E_1 of Example 6.5. The (graded) ideal I generated by the hereditary saturated set $H_1=\{w,w_1,w_2,u_1,u_2,u_3\}$ is, by Theorem 7.2, a minimal prime ideal of $L_K(E_3)$, since $S=\{w,w_1,w_2\}$ is the only proper maximal tail in H_1 and for $w_1\in S$ and $v\notin H_1$, we have $T(w_1)\cap T(v)=\Phi$.

For row-finite graphs Theorem 7.2 reduces to the following.

Corollary 7.3. *Let E be a row-finite graph and K be any field. Then a non-zero prime ideal P with $P\cap E^0=H$ is a minimal prime ideal if and only if $P=\langle H\rangle$ and H is satisfies the Property $(*)$.*

8. Height one prime ideals of $L_K(E)$

Recall that the **height** of a prime ideal P is n if n is the largest integer such that there exists a chain of different prime ideals $P=P_0\supsetneqq P_1\supsetneqq\cdots\supsetneqq P_n$. Thus minimal prime ideals have height 0. Also a ring R has Krull dimension 0 if every prime ideal of R has height 0. The concept the height can also be defined for infinite ordinals λ in an analogous fashion. The **height one** prime ideals play an important role in the study of commutative rings and algebraic geometry. In this section, we describe the height one prime ideals of an arbitrary Leavitt path algebra.

Theorem 8.1. *Let E be an arbitrary graph and K be any field. Then a prime ideal P of the Leavitt path algebra $L_K(E)$ with $P\cap E^0=H$ is a prime ideal of height one if and only if P has one of the following properties*:

(i) *P is a non-graded prime ideal with either $H=\Phi$ (the empty set) or H satisfies Property $(*)$ and for every $u\in B_H$, $M(u)\neq E^0\backslash H$;*
(ii) *$P=I_{(H,B_H)}$ and, either there is a vertex $u\in B_H$ such that $M(u)=E^0\backslash H$ and H satisfies Property $(*)$ in E^0 or $M(u)\neq E^0\backslash H$ for any $u\in B_H$ and there is exactly one maximal tail in E^0 properly containing $E^0\backslash H$;*
(iii) *$P=I_{(H,B_H\backslash\{u\})}$ and there is exactly one maximal tail in E^0 properly containing $M(u)$.*

Proof. We distinguish three cases corresponding to the three types of prime ideals indicated in Theorem 3.12.

Suppose P is a non-graded prime ideal. Now, by Lemmas 3.6 and 3.8, $I_{(H,B_H)}$ is a prime ideal that contains every graded ideal inside P. Moreover, $I_{(H,B_H)}$ also contains any non-graded prime ideal of $L_K(E)$ inside P. Because, if $J=\langle I_{(H',B_{H'})},g(c')\rangle$ is a non-graded prime ideal properly contained in P with c' a cycle without exits based at a vertex v' in $E^0\backslash H'$, then necessarily $H'\subsetneqq H$ and so H contains a $w\in E^0\backslash H'=M(v')$. Then the hereditary set H contains v' and hence $(c')^0$ and this implies that the ideal $I_{(H,B_H)}$ contains $g(c')$ and hence J. Consequently, P has height one if and only if $I_{(H,B_H)}$ is a minimal prime ideal. By Theorem 7.2, this is possible if and only if either $I_{(H,B_H)}=\{0\}$, that is $H=\Phi$, or for every $u\in B_H$, $M(u)\neq E^0\backslash H$ and H satisfies Property $(*)$.

Suppose P is a graded prime ideal of the form $I_{(H,B_H)}$. By Theorem 3.12, $I_{(H,B_H)}$ contains a prime ideal of the form $I_{(H,B_H\setminus\{u\})}$ for some $u \in B_H$ exactly when $M(u) = E^0\setminus H$. Thus in this case, $I_{(H,B_H)}$ will have height one if and only if $I_{(H,B_H\setminus\{u\})}$ is a minimal prime ideal. Appealing to Theorem 7.2, we then conclude that in this case $I_{(H,B_H)}$ has height one if and only if for some $u \in B_H$, $M(u) = E^0\setminus H$ and H satisfies the Property $(*)$. Suppose on the other hand, $M(u) \neq E^0\setminus H$ for any $u \in B_H$. If $I_{(H,B_H)}$ has height one, then the unique prime ideal $J \subsetneqq I_{(H,B_H)}$ must necessarily be a graded ideal, since every non-graded prime ideal, by Lemma 3.8, contains another (graded) prime ideal. Moreover, $X = J \cap E^0 \subsetneqq H$ since otherwise $J = I_{(H,B_H\setminus\{u\})}$ for some $u \in B_H$ and this contradicts our supposition that $M(u) \neq E^0\setminus H$ for any $u \in B_H$. Thus the graded ideal J is the only prime ideal contained in $I_{(H,B_H)}$ with $X = J \cap E^0 \subsetneqq H$ and this happens if and only if $M = E^0\setminus X$ is the only maximal tail properly containing $E^0\setminus H$.

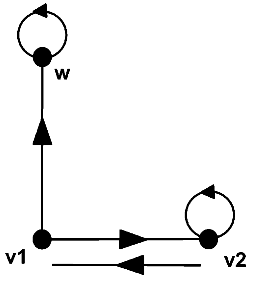
Suppose P is a graded prime ideal of the form $I_{(H,B_H\setminus\{u\})}$ for some $u \in B_H$. Then $I_{(H,B_H\setminus\{u\})}$ has height one if and only if there exists exactly one prime ideal $J \subsetneqq P$. As argued in the preceding paragraph, this ideal J has to be graded ideal with $J \cap E^0 = X \subsetneqq H$. Thus J will be the only prime ideal contained in P if and only if $M = E^0\setminus X$ is the only maximal tail properly containing $E^0\setminus H$. $\square$

Remark. From the proof of Theorem 8.1 we get a slightly sharper form of Lemmas 3.6 and 3.8 for prime ideals: If $P = \langle I_{(H,B_H)}, f(c)\rangle$ is a non-graded prime ideal of $L_K(E)$, then $I_{(H,B_H)}$ is a prime ideal that contains every other prime ideal of $L_K(E)$ inside P and also contains every graded ideal of $L_K(E)$ inside P.

For row-finite graphs we get the following.

Corollary 8.2. *Let E be a row-finite graph and K be any field. Then a prime ideal P of $L_K(E)$ with $P \cap E^0 = H$ has height* 1 *if and only if either P is a non-graded ideal with H empty or with H satisfying the Property* $(*)$*, or P is a graded ideal such that there is exactly one maximal tail containing $E^0\setminus H$.*

Example. Consider the graph G given below:



Let c denote the loop based at w. Now G^0 satisfies the MT-3 condition and so $\{0\}$ is a prime ideal of $L_K(G)$, by Theorem 3.1. Note that c is a cycle without exits in G. Now the non-zero proper ideals of $L_K(G)$ are the graded ideal $\langle w\rangle$ and the infinitely many non-graded ideals $\langle p(c)\rangle$, for various irreducible polynomials $p(x) \in K[x, x^{-1}]$. By the above theorem, every non-zero proper ideal of $L_K(G)$ is a prime ideal of height one.

9. Co-height one prime ideals of $L_K(E)$

A prime ideal P of a ring R is said to have **co-height** n if n is the largest integer such that there exists a chain of different prime ideals $P = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n \neq R$. Thus, in particular, P will have **co-height** 1 if there exists a prime ideal $Q \neq R$ such that $P \subset Q$ and there is no prime ideal I such that $P \subsetneqq I \subsetneqq Q$ and no prime ideal J such that $Q \subsetneqq J \subsetneqq R$ and every other prime ideal $P' \supset P$ also has the same property as Q.

We shall describe the prime ideals of co-height 1 by means of graphical properties of E and construct examples illustrating these properties.

Theorem 9.1. *Let E be an arbitrary graph and K be any field. Then a prime ideal P of the Leavitt path algebra $L_K(E)$ with $P \cap E^0 = H$ has co-height* 1 *if and only if P satisfies one of the following conditions*:

(i) *P is a non-graded ideal such that all the non-empty maximal tails properly contained in $E^0 \backslash H$ satisfy the Condition* (L) *and form a non-empty antichain under set inclusion*;
(ii) *P is a graded ideal of the form $I_{(H,B_H)}$ such that either* (a) *all the non-empty maximal tails properly contained in $E^0 \backslash H$ satisfy the Condition* (L) *and form a non-empty antichain under set inclusion or* (b) *there are no non-empty maximal tails properly contained in $E^0 \backslash H$ and there exists a cycle without exits in $E^0 \backslash H$ based at a vertex v such that $E^0 \backslash H = M(v)$*;
(iii) *P is a graded ideal of the form $I_{(H,B_H \backslash \{u\})}$ for some $u \in B_H$, such that every cycle containing u has exits in $E^0 \backslash H$ and that there are no non-empty maximal tails properly contained in $E^0 \backslash H$.*

Proof. By Theorem 3.12, there are three types prime ideals in $L_K(E)$. Accordingly, we consider three different cases.

Case (i): Let P be a non-graded prime ideal, so by Theorem 3.12, $P = \langle I_{(H,B_H)}, f(c)\rangle$ where c is a cycle without exits based at a vertex v in $E^0 \backslash H$ and $E^0 \backslash H = M(v)$.

Suppose P has co-height 1. So if $P \subset Q$ for some prime ideal Q, then there is no prime ideal I such that $P \subsetneqq I \subsetneqq Q$ and no prime ideal J such that $Q \subsetneqq J \subsetneqq L_K(E)$ and that there is at least one such prime ideal Q. We claim that Q cannot be a non-graded prime ideal. Suppose, on the contrary, $Q = \langle I_{(H',B_{H'})}, g(c')\rangle$ where $H' = Q \cap E^0$, c' is a cycle without exits in $E^0 \backslash H'$. By the remark preceding Corollary 8.2, $I_{(H',B_{H'})}$ properly contain P, contradicting the fact that there is no prime ideal I with $P \subsetneqq I \subsetneqq Q$. Thus every prime ideal $Q \supsetneqq P$ must be a graded prime ideal, say, $Q = I_{(H',B_{H'})}$ with $H' \supsetneqq H$. Clearly $v \in H'$. Now $M = E^0 \backslash H'$ is a maximal tail properly contained in $E^0 \backslash H$. We claim that M satisfies the Condition (L). Because, otherwise, there will be a cycle c'' without exits in M based at a vertex v' and $M = M(v')$. Then for any irreducible polynomial $p(x) \in K[x, x^{-1}]$, the ideal $\langle Q, p(c'')\rangle$ will be, by Theorem 3.12, a prime ideal containing Q, a contradiction. If there is a maximal tail N inside $E^0 \backslash H$ such that $M \subsetneqq N$ then since $H \subset X = E^0 \backslash N \subset H'$, we have $P \subsetneqq I = I_{(X,B_X)} \subsetneqq Q$, a contradiction. If, on the other hand, there is a non-empty maximal tail $N' \subsetneqq M$ then for $Y = E^0 \backslash N'$, the ideal $J = I_{(Y,B_Y)}$ is a prime ideal satisfying $Q \subsetneqq J \subsetneqq L_K(E)$, a contradiction. Hence the non-empty maximal tails properly contained in $E^0 \backslash H$ form an antichain under set inclusion and satisfy the Condition (L). Note that there is at least one such maximal tail in $E^0 \backslash H$, since there is at least one prime ideal $Q \supsetneqq P$.

Conversely, suppose P satisfies Condition (i) of the theorem. By hypothesis, the set S of non-empty maximal tails properly contained in $E^0 \backslash H$ is non-empty. Let M be an arbitrary member of this set S and let $H' = E^0 \backslash M$. Then $Q = I_{(H',B_{H'})}$ is a prime ideal containing P. If there is a prime ideal I such that $P \subsetneqq I \subsetneqq Q$ then for $X = I \cap E^0$, we have $H \subsetneqq X \subsetneqq H'$ and so $N = E^0 \backslash X$ is a maximal tail that satisfies $M \subsetneqq N \subsetneqq E^0 \backslash H$, thus contradicting the hypothesis. Suppose there is a prime ideal J with $Q \subsetneqq J \subsetneqq L_K(E)$. Let $J \cap E^0 = Y$. We claim $Y \neq H'$. Indeed if $Y = H'$, then first of all J cannot be a graded ideal since then $B_Y = B_{H'}$ and $J = I_{(Y,B_Y)} = I_{(H',B_{H'})} = Q$, a contradiction. On the other hand if J were a non-graded prime ideal, then there must be a cycle without exits in $E^0 \backslash Y = E^0 \backslash H'$, again a contradiction since $E^0 \backslash H'$ satisfies Condition (L). Thus $Y \supsetneqq H'$. But then the maximal tail $E^0 \backslash Y$ satisfies $E^0 \backslash Y \subsetneqq M$, a contradiction to the hypothesis. This shows that P is a prime ideal of co-height 1.

Case (ii): Let P be a graded prime ideal of the form $I_{(H,B_H)}$.

Suppose P has co-height 1, so there is a prime ideal $Q \supset P$ such that there is no prime ideal I such that $P \subsetneqq I \subsetneqq Q$ and no prime ideal J such that $Q \subsetneqq J \subsetneqq L_K(E)$. Let $Q \cap E^0 = H'$.

If Q is a non-graded prime ideal, say, $Q = \langle I_{(H',B_{H'})}, p(c)\rangle$, then necessarily, $H' = H$ since otherwise, by Lemma 3.6, $I_{(H',B_{H'})}$ will be a prime ideal that satisfies $P \subsetneqq I_{(H',B_{H'})} \subsetneqq Q$. Thus $Q =$

$\langle I_{(H,B_H)}, p(c)\rangle$, with c a cycle without exits in $E^0\backslash H$ based at a vertex v and $E^0\backslash H = M(v)$. Also $E^0\backslash H$ cannot contain any proper non-empty maximal tail M since otherwise $X = E^0\backslash M$ will be a hereditary saturated subset containing v and hence c^0, and so the ideal $I_{(X,B_X)}$ is a prime ideal containing Q, a contradiction. This shows that P satisfies Condition (ii) (b).

Suppose Q be a graded prime ideal. By Theorem 3.12, $Q = I_{(H',B_{H'})}$ or $I_{(H',B_{H'}\backslash\{u\})}$ for some $u \in B_{H'}$. Since $I_{(H',B_{H'})} \supsetneqq I_{(H',B_{H'}\backslash\{u\})}$ and there is no prime ideal properly containing Q, $Q \neq I_{(H',B_{H'}\backslash\{u\})}$. Thus $Q = I_{(H',B_{H'})}$ and $H' \neq H$ since, otherwise, $Q = I_{(H,B_H)} = P$, a contradiction. Now $M = E^0\backslash H'$ is a maximal tail properly contained in $E^0\backslash H$. If N is a non-empty maximal tail and $N \subsetneqq M$ and $X = E^0\backslash N$, then $I_{(X,B_X)}$ will be a prime ideal satisfying $Q \subsetneqq I_{(X,B_X)} \subsetneqq L_K(E)$, a contradiction. Also, if there is a maximal tail N' with $M \subsetneqq N' \subsetneqq E^0\backslash H$, then for $Y = E^0\backslash N'$, the prime ideal $I_{(Y,B_Y)}$ satisfies $P \subsetneqq I_{(Y,B_Y)} \subsetneqq Q$, a contradiction. Also, if the maximal tail M does not satisfy Condition (L), then there will be a cycle without exits based at a vertex v in M so that $M = M(v)$. Then for an irreducible polynomial $f(x) \in K[x, x^{-1}]$, the prime ideal $\langle Q, f(c)\rangle$ properly contains Q, a contradiction. Hence M satisfies Condition (L). We thus have shown that P satisfies Condition (ii) (a).

Conversely, suppose the prime ideal $P = I_{(H,B_H)}$ satisfies the stated properties in Condition (ii) of the theorem.

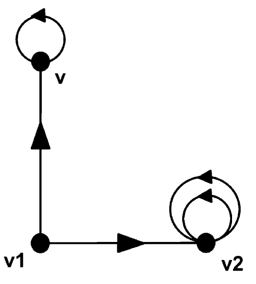
Specifically, assume Condition (ii) (b) so that there is a cycle c without exits in $E^0\backslash H$ based at a vertex v, $E^0\backslash H = M(v)$ and $E^0\backslash H$ contains no non-empty proper maximal tails. Then for some irreducible polynomial $f(x) \in K[x, x^{-1}]$, $Q = \langle I_{(H,B_H)}, f(c)\rangle$ is a prime ideal containing P. Clearly, there is no ideal I such that $P \subsetneqq I \subsetneqq Q$. Now there cannot be a prime ideal J, with $Q \subsetneqq J \subsetneqq L_K(E)$. Because, if $Y = J \cap E^0$, first of all $Y \neq H$ since, otherwise, J has to be a non-graded prime ideal of the form $J = \langle I_{(H,B_H)}, g(c)\rangle$ for some irreducible polynomial $g(x) \in K[x, x^{-1}]$ and this is impossible by the remark at the end of Section 3. But if $Y \supsetneqq H$ then $M = E^0\backslash Y$ will be a maximal tail properly contained in $E^0\backslash H$, a contradiction. This shows that P has co-height 1.

Suppose now that Condition (ii) (a) holds. By hypothesis, the set S of non-empty maximal tails properly contained in $E^0\backslash H$ is non-empty. Now repeat the proof of the converse in Case (i) above verbatim to conclude that P has co-height 1.

Case (iii): Let P be a graded prime ideal of the form $P = I_{(H,B_H\backslash\{u\})}$. Since $E^0\backslash H = M(u)$ satisfies the MT-3 condition, $I_{(H,B_H)}$ will always be a prime ideal containing P. It is readily seen that there is no prime ideal I with $I_{(H,B_H\backslash\{u\})} \subsetneqq I \subsetneqq I_{(H,B_H)}$. So P will have co-height 1 exactly when there are no prime ideals $P' \supsetneqq I_{(H,B_H)}$. If a prime ideal $P' \supsetneqq I_{(H,B_H)}$ with $X = P' \cap E^0$, then either $X = H$ or $X \supsetneqq H$. If $X = H$, then P' must be non-graded and so, by Theorem 3.12, $E^0\backslash H = E^0\backslash X$ will have a cycle without exits. This is possible if and only if $E^0\backslash H$ does not satisfy Condition (L). On the other hand, $X \supsetneqq H$ if and only if $E^0\backslash X$ is a proper non-empty maximal tail in $E^0\backslash H$. This proves that $I_{(H,B_H\backslash\{u\})}$ has co-height 1 if and only if $E^0\backslash H$ satisfies the Condition (L) and contains no non-empty proper maximal tails. □

Example 9.2.

1. Let E be the graph.



Let c denotes the loop at the vertex v. Clearly E^0 satisfies the MT-3 condition and so, by Theorem 3.12, the ideal $P = \langle v - c\rangle$ is a non-graded prime ideal of $L_K(E)$. Now $H = \{u, v\}$

is a hereditary saturated subset and $P \subset Q = \langle H \rangle$ which is a maximal ideal of $L_K(E)$, since $L_K(E)/Q \cong L_K(E^0 \backslash H) \cong L(1,2)$, a Leavitt algebra which is a simple ring (see [1]). Thus P is a co-height one non-graded prime ideal. Note that Condition (i) of the above theorem is trivially satisfied.
2. In Example 4.6, the ideal $Q = I_{(H,B_H)}$ is the only co-height 1 prime ideal of $L_K(E)$ and is graded. Note that Condition (ii) of the above theorem holds for $E^0 \backslash H$.
3. Consider Example 4.5. For each $j \geqslant 1$, the prime ideal P_j has co-height 1. Note that Condition (iii) of the above theorem holds. It is interesting to note that each P_j is also a height 1 prime ideal.

10. Prime homomorphic images of $L_K(E)$

Finally, we consider the prime homomorphic images of a Leavitt path algebra $L_K(E)$. Should they all be again Leavitt path algebras? Fruitful correspondence with Ken Goodearl resulted in a definitive answer to this question in the case of a finite graph E and this appears as Proposition 4.4 in [4]. As an application of Theorem 3.12, we get a complete description of the prime homomorphic images of $L_K(E)$ for arbitrary graphs E. The same proof shown in [4], with minor modifications, works for arbitrary graphs E. We outline the proof for the sake of completeness.

Proposition 10.1. *Let E be an arbitrary graph and let P be a prime ideal of $L_K(E)$. Then either $T = L_K(E)/P$ is isomorphic to a Leavitt path algebra or $T/Soc(T)$ is isomorphic to a Leavitt path algebra. In the latter case, $Soc(T)$ is a simple ring being a direct sum of isomorphic simple left ideals of T.*

Proof. If $P = I_{(H,S)}$ is a graded ideal, then, by [22], $T = L_K(E)/P \cong L_K(E \backslash (H,S))$. Suppose P is a non-graded prime ideal. By Theorem 3.12, $P = \langle I_{(H,B_H)}, f(c) \rangle$ where c is a unique cycle without K in E based at a vertex v and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. Note that $v \notin B_H$. As pointed out in the proof of Theorem 4.3 (i), for the idempotent $\bar{v} = v + P$, we have $\bar{v} T \bar{v}$ is a field. This implies that $T\bar{v}$ is then a simple left ideal of T by Proposition 1, Chapter 4, p. 65 in [17] (observing that the prime ring T has no non-zero nilpotent one-sided ideals). Thus $T\bar{v} \subset Soc(T)$. Since T is a prime ring, its socle is a direct sum of isomorphic simple left ideals and, in particular, is the two-sided ideal generated by any simple left ideal in it. We then conclude that $Soc(T) = \langle \bar{v} \rangle = (\langle v \rangle + P)/P$, the ideal generated by $\bar{v}$. Now $P + \langle v \rangle = I_{(H,B_H)} + \langle v \rangle = J$ is a graded ideal (being the sum of two graded ideals). Thus $T/Soc(T) = (L_K(E)/P)/(P + \langle v \rangle/P) \cong L_K(E)/(P + \langle v \rangle)$ which is isomorphic to the Leavitt path algebra $L_K(E \backslash (H', S'))$ where $H' = J \cap E^0$ and $S' = \{w \in B_{H'}:\ w^{H'} \in J\}$. $\square$

Remark 10.2. In the above proposition, if E^0 (or more generally, $(E \backslash H, S)^0$) is countable, then $Soc(T)$ will be a direct sum of countably many isomorphic simple left ideals and, in this case, $Soc(T) \cong L_K(F)$ where F is the infinite straight line graph $\overset{v_1}{\bullet} \to \overset{v_2}{\bullet} \to \overset{v_3}{\bullet} \to \bullet \to \bullet \cdots$. So T can then be realized as an extension of a Leavitt path algebra by another Leavitt path algebra. Moreover, define a new graph G by forming the disjoint union of the graphs F and $E \backslash (H', S')$ (which was defined in the proof of Proposition 10.1) and connecting each line point in $E \backslash (H', S')$ to the vertex v_1 of the graph F by an edge. Then in the graph G, the line points are precisely the vertices of the graph F and the quotient graph $G \backslash F$ is the same as the graph $E \backslash (H', S')$. Then the Leavitt path algebra $L_K(G)$ has the property that $Soc(L_K(G)) \cong Soc(T)$ and $L_K(G)/Soc(L_K(G)) \cong T/Soc(T)$.

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References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2005) 319–334.

[2] G. Abrams, G. Aranda Pino, The Leavitt path algebras of arbitrary graphs, Houston J. Math. 34 (2008) 423–442.
[3] G. Abrams, G. Aranda Pino, M. Siles Molina, Locally finite Leavitt path algebras, Israel J. Math. 165 (2008) 329–348.
[4] G. Abrams, J. Bell, K.M. Rangaswamy, The Diximier–Moeglin equivalence for Leavitt path algebras, Algebr. Represent. Theory 15 (2012) 407–425.
[5] G. Abrams, J. Bell, K.M. Rangaswamy, On prime non-primitive von Neumann algebras, Trans. Amer. Math. Soc., in press.
[6] G. Abrams, J. Bell, P. Colak, K.M. Rangaswamy, Two-sided chain conditions in Leavitt path algebras over arbitrary-sized graphs, J. Algebra Appl. 11 (2012).
[7] G. Abrams, K.M. Rangaswamy, M. Siles Molina, Socle series of Leavitt path algebras of arbitrary graphs, Israel J. Math. 184 (2011) 413–435.
[8] G. Abrams, K.M. Rangaswamy, Regularity conditions for arbitrary Leavitt path algebras, Algebr. Represent. Theory 13 (2010) 319–334.
[9] P. Ara, M.A. Moreno, E. Pardo, Non-stable K-theory for graph algebras, Algebr. Represent. Theory 10 (2007) 157–178.
[10] P. Ara, E. Pardo, Stable rank of Leavitt path algebras, Proc. Amer. Math. Soc. 136 (2008) 2375–2386.
[11] G. Aranda Pino, E. Pardo, M. Siles Molina, Prime spectrum and primitive Leavitt path algebras, Indiana Univ. Math. J. 58 (2009) 869–890.
[12] G. Aranda Pino, D. Marti'n Barquero, C. Marti'n Gonzalez, M. Siles Molina, Socle theory for Leavitt path algebras of arbitrary graphs, Rev. Mat. Iberoamericana 26 (2) (2010) 611–638.
[13] G. Aranda Pino, E. Pardo, M. Siles Molina, Exchange Leavitt path algebras and stable rank, J. Algebra 305 (2) (2006) 912–936.
[14] J. Dixmier, Sur les C^*-algèbres, Bull. Soc. Math. France 88 (1960) 95–112.
[15] K. Goodearl, Leavitt path algebras and direct limits, in: Contemp. Math., vol. 480, Amer. Math. Soc., Providence, RI, 2009, pp. 165–187.
[16] J.H. Hong, W. Szymański, The primitive ideal space of the C^*-algebras of infinite graphs, J. Math. Soc. Japan 56 (2004) 45–64.
[17] N. Jacobson, Structure of Rings, revised edition, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, RI, 1964.
[18] T.Y. Lam, A First Course in Non-Commutative Rings, Springer-Verlag, New York, NY, 2001.
[19] C. Lanski, R. Resco, L. Small, On the primitivity of prime rings, J. Algebra 59 (1979) 395–398.
[20] C. Năstăcescu, F. Van Oystaeyen, Graded Ring Theory, North-Holland, Amsterdam, 1982.
[21] I. Raeburn, Graph Algebras, CBMS Reg. Conf. Ser. Math., vol. 103, Amer. Math. Soc., Providence, RI, 2005.
[22] M. Tomforde, Uniqueness theorems and ideal structure of Leavitt path algebras, J. Algebra 318 (2007) 270–299.